A geometric derivation of the linear Boltzmann equation for a particle interacting with a Gaussian random field

Sébastien Breteaux

A GEOMETRIC DERIVATION OF THE LINEAR BOLTZMANN EQUATION FOR A PARTICLE INTERACTING WITH A GAUSSIAN RANDOM FIELD

SÉBASTIEN BRETEAUX

Abstract. In this article the linear Boltzmann equation is derived for a particle interacting with a Gaussian random field, in the weak coupling limit, with renewal in time of the random field. The initial data can be chosen arbitrarily. The proof is geometric and involves coherent states and semi-classical calculus.

1. Introduction

In this article we derive the linear Boltzmann equation for a particle interacting with a translation invariant centered Gaussian random field. The evolution of this particle is described by the Liouville - Von Neumann equation with a Hamiltonian $-\Delta_x + \mathcal{V}_\omega^h(x)$, where the potential depends on a random parameter ω. In the weak coupling limit, the dependence of the random potential with respect to h is $\mathcal{V}_\omega^h = \sqrt{h}\mathcal{V}_\omega$, where h represents the ratio between the microscopic and macroscopic scales. We consider the limit $h \to 0$. In the case of a Gaussian random field the weak coupling limit and the low density limit agree. Through an isomorphism between the Gaussian space $L^2(\Omega_\mathbb{P}, \mathbb{P}; \mathbb{C})$ associated with $L^2(\mathbb{R}^d; \mathbb{R})$ and the symmetric Fock space $\Gamma L^2(\mathbb{R}^d)$ associated with $L^2(\mathbb{R}^d; \mathbb{C})$, multiplication by $\mathcal{V}_\omega(x)$ corresponds to the field operator $\sqrt{2}\Phi(V(x-\cdot))$ for some function V. We can thus express the Hamiltonian in the Fock space and approximate the dynamics by an explicitly solvable one whose solutions are coherent states. The geometric idea behind the computations is due to the fact that the initial state is the vacuum, and we can thus expect that for short times the system is approximately in a coherent state whose parameter moves slightly in the phase space. This parameter in the (infinite dimensional) phase space then gives the important information in the limit $h \to 0$. The computations done with this solution allow us to recover the dual linear Boltzmann equation for short times for the observables. A renewal of the random field allows us to reach long times.

The derivation of the linear Boltzmann equation has been studied for both classical and quantum microscopic models. In the classical case Gallavotti [18] provided a derivation of the linear Boltzmann equation for Green functions in the case of a Lorentz gas. Later Spohn [37] presented a review of different classical microscopic models and of kinetic equations obtained as limits of these models, with emphasis on the approximate Markovian behaviour of the microscopic dynamics (some quantum models were also studied). Boldrighini, Bunimovich and Sinaĭ [8] gave a derivation of the linear Boltzmann equation for the density of particles in

2000 *Mathematics Subject Classification.* 82C10, (60K37, 81E, 81S, 81D30, 82B44, 82C40).

Key words and phrases. Linear Boltzmann equation, Processes in random environments, Quantum field theory, Coherent states, Kinetic theory of gases.

the case of the Lorentz model. In the quantum case, Spohn derived in [36] the radiative transport equation in the spatially homogenous case. Later Ho, Landau and Wilkins studied in [29] the weak coupling limit of a Fermi gas in a translation invariant Gaussian potential (and other random potentials). Their proofs made use of combinatorics and graph techniques. In the case of a particle interacting with a Gaussian random field (the setting of this article) Erdős and Yau [16] removed the small time restriction, and also generalized the initial data to WKB states, using methods with graph expansions. Developements of that method by Chen [11] and Erdős, Salmhofer and Yau [15, 14] did not require a Gaussian form for the random field but still supposed an initial state of the WKB form. The linear Boltzmann equation was derived in the radiative transport limit by Bal, Papanicolaou and Ryzhik [5] in the quantum case, and by Poupaud and Vasseur [32] in the classical case using a potential stochastic in time. This assumption automatically ensures that there is no self-correlation in the paths of the particles and simplifies the problem. Later Bechouche, Poupaud and Soler [6] used similar techniques to get a model for collisions at the quantum level and obtain a kind of quantum linear Boltzmann equation. For these stochastic methods the initial state can be arbitrary but the potential is almost surely bounded, which excludes Gaussian or Poissonian random fields.

Remarks. Our derivation is given in the case of a Gaussian random field but other random fields could be considered with the same type of methods, for example a Poissonian random field. Note that the weak coupling and low density limit do not then agree.

Our approach allows initial states to be arbitrary, contrary to WKB initial states.

The framework of quantum field theory allows to see how geometry in phase space is involved. We use the viewpoint of Ammari and Nier [1] but in a case that is not in the framework chosen by the authors. Indeed we are not dealing with a mean field limit and the introduction of a parameter ε is an artifact that allows us to keep track of the importance of the different terms. We thus adopt a different viewpoint from the graph expansions or the stochastic viewpoint adopted in other works on the subject, and this allows us to keep track of the geometry.

However, we cannot as of yet reach times of order 1 like in [16, 11, 14, 15]. As we do not get the approximate Markovian behaviour in a satisfying way, we need to introduce a renewal of the random potential. Attal and Pautrat in [4] and Attal and Joye in [3] deal in a more sophisticated way with interactions defined piecewise in time. Other Ansätze may give a better approximation of the solution to the initial problem and give the Markovian behaviour of the evolution.

One of the important tools in our derivation of the linear Boltzmann equation is the use of *a priori* estimates to show that we do not lose too much mass in the measures during our approximations. The mass conservation and positivity properties of the linear Boltzmann equation then allow us to complete the proof.

Our result holds in dimension $d \geq 3$ as dispersion inequalities for the free Schrödinger group provide the time integrability needed for some expressions. It may be possible to reach the limit case of dimension $d = 2$.

Outline of the article. In Section 2, we describe the quantum model, state the main result and give the structure of the proof. We then recall some facts about the linear Boltzmann equation in Section 3. We specify the link between the Gaussian random field and the symmetric Fock space in Section 4 and thus obtain a new expression for the dynamics. We study an approximate dynamics in Section 5. We use this explicit solution to compute the measurement of an observable for short times in Section 6. We control the error involved in this approximation in Section 7. And finally, we combine these results to complete the proof in Section 8.

2. Model and result

2.1. The model. Let $\omega \in \Omega_{\mathbb{P}}$ be a random parameter and $x \in \mathbb{R}^d$ $(d \geq 1)$ a space parameter. Let $\mathcal{V}^h_\omega(x)$ the translation invariant centered Gaussian random field with mean zero and covariance $hG(x-x')$, such that $\hat{G} = |\hat{V}|^2$ with $\hat{V} \in \mathcal{S}\left(\mathbb{R}^d;\mathbb{R}\right)$. We consider the Liouville - Von Neumann equation

$$ih\partial_t \rho_{t,\omega} = [H^h_\omega, \rho_{t,\omega}]\,, \qquad H^h_\omega = -\Delta_x + \mathcal{V}^h_\omega(x)\,, \tag{2.1}$$

with an initial condition $\rho^h_{0,\omega} = \rho^h_0$ in the set of states on $L^2_x = L^2(\mathbb{R}^d_x;\mathbb{C})$ (*i.e.* the subset of the non-negative trace class operators $\mathcal{L}^+_1(L^2_x)$ whose trace is 1). Note that $[A,B]$ denotes the commutator $AB - BA$ of two operators.

We now introduce the renewal of the random field. We fix a time T, an integer N and set $\Delta t = T/N$. For a state ρ on L^2_x, let

$$\mathcal{G}^h_t(\rho) = \int e^{-i\frac{t}{h}H_{h,\omega}}\,\rho\, e^{i\frac{t}{h}H_{h,\omega}}\,\mathrm{d}\mathbb{P}(\omega)\,, \tag{2.2}$$

$$\rho^h_t = \mathcal{G}^h_t(\rho)\,, \tag{2.3}$$

$$\rho^h_{N,\Delta t} = (\mathcal{G}^h_{\Delta t})^N(\rho)\,. \tag{2.4}$$

With $t_k = k\Delta t$, the dynamics is defined piecewise on the intervals $[t_{k-1}, t_k]$ by the Hamiltonians $H_{h,\omega_k} = -\Delta_x + \mathcal{V}_{h,\omega_k}(x)$ with independent random fields $\mathcal{V}_{h,\omega_k}$, ω_k in copies of $\Omega_{\mathbb{P}}$. Thus we get, for an initial data $\rho_0 \in \mathcal{L}^1_+(L^2_x)$, that the system is in the state $\rho^h_{N,\Delta t}$ at time T.

2.2. The main result. Let $b \in \mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$. The measure of the observable $b^W(hx, D_x)$ in a state ρ on L^2_x is given by

$$m_h(b,\rho) = \mathrm{Tr}\big[b^W(hx,D_x)\rho\big]\,,$$

where the *Weyl quantization* (see for example Martinez's book [31]) is defined by

$$b^W(hx,D_x)u\,(x) = (2\pi)^{-d}\int_{\mathbb{R}^{2d}_{x',\xi}} e^{i(x-x')\cdot\xi}\, b\big(h\tfrac{x+x'}{2},\xi\big)\,u(x')\,\mathrm{d}x'\,\mathrm{d}\xi\,.$$

Semiclassical measures (and microlocal defect measures) have been studied by, among others, Gérard [19, 20], Burq [10], Gérard, Markowich, Mauser and Poupaud [21, 22] and Lions and Paul [30]. Let us quote Theorem 2.1, which is a direct consequence of a theorem which can be found in [10] (with (ρ^h) replacing $(|u_k\rangle\langle u_k|)$ for weakly convergent sequence (u_k) of L^2_x).

Theorem 2.1. *Let $(\rho^h)_{h\in(0,h_0]}$, $h_0 > 0$ be a family of states on L^2_x. There exist a sequence $h_k \to 0$ and a non-negative measure μ on $\mathbb{R}^{2d}_{x,\xi}$ such that*

$$\forall b \in \mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})\,, \quad \lim_{n\to+\infty} m_{h_k}(b,\rho^{h_k}) = \int_{\mathbb{R}^{2d}_{x,\xi}} b\;\mathrm{d}\mu\,.$$

The measure μ is called a semiclassical measure *(or Wigner measure) associated with the sequence (ρ^{h_k}). Let $\mathcal{M}(\rho^h, h \in (0, h_0])$ be the set of such measures. If this set is a singleton $\{\mu\}$ then the family (ρ^h) is said to be* pure *and associated with μ.*

By a simple sequence extraction out of the range of the parameter h, the family can always be assumed to be pure. For evolution problems the fact that the sequence extraction can be performed uniformly for all times is a property to be proved.

We can now state the main theorem of this article.

Theorem 2.2. *Assume $d \geq 3$. Let $\Delta t = h^\alpha$, $N = N^h = T/h^\alpha$, $\alpha \in (\frac{3}{4}, 1)$.*

Assume that $(\rho^h)_{h\in(0,h_0]}$ is pure and associated with μ_0 such that $\mu_0(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi) = 1$.

Then $(\rho^h_{N,\Delta t})_{h\in(0,h_0]}$ is pure and associated with μ_T, where $(\mu_t)_t$ solves the linear Boltzmann equation

$$\partial_t \mu_t(x,\xi) + 2\xi.\partial_x \mu_t(x,\xi) = \int \sigma(\xi,\xi')\,\delta\big(|\xi|^2 - |\xi'|^2\big)\big(\mu_t(x,\xi') - \mu_t(x,\xi)\big)\,\mathrm{d}\xi' \tag{2.5}$$

with the initial condition $\mu_{t=0} = \mu_0$ and $\sigma(\xi,\xi') = 2\pi|\hat{V}(\xi-\xi')|^2$.

The Fourier transform on $\mathbb{R}^d$ is here $\hat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^d_x} e^{-ix.\xi} u(x)\,\mathrm{d}x$.

Sketch of the Proof. Let μ_T in $\mathcal{M}(\rho^h_{N,\Delta t}, h \in (0, h_0])$. We denote by $\mathcal{B}(t)$ (resp. $\mathcal{B}^T(t)$) the flow associated with the (resp. dual) Boltzmann equation (2.5), see Section 3. For any non-negative b in $\mathcal{C}^\infty_0(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$ we shall prove

1. $\int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi} b\,\mathrm{d}\mu_T \geq \liminf_{h\to 0} \mathrm{Tr}[\rho^h_{N,\Delta t}\, b^W(hx, D_x)]$ by the definition of μ_T,
2. $\liminf_{h\to 0} \mathrm{Tr}[\rho^h_{N,\Delta t}\, b^W(hx, D_x)] \geq \int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi} (\mathcal{B}^T(T)b)\,\mathrm{d}\mu_0$ (see Remark 2.3),
3. $\int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi} (\mathcal{B}^T(T)b)\,\mathrm{d}\mu_0 = \int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi} b\ \mathrm{d}(\mathcal{B}(T)\mu_0)$ by the definition of $\mathcal{B}(T)$.

From these statements, the lower bound

$$\int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi} b\ \mathrm{d}\mu_T \geq \int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi} b\ \mathrm{d}(\mathcal{B}(T)\mu_0)$$

follows. Since this inequality holds for any non-negative b from the set of smooth functions with compact support $\mathcal{C}^\infty_0(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$, which is dense in the set of continuous functions vanishing at "infinity" $\mathcal{C}^0_\infty(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$, whose dual is the set of Radon measures $\mathcal{M}_b(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$, we get

$$\mu_T\big|_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi} \geq \mathcal{B}(T)\mu_0\big|_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi}\ .$$

But we also have $\mathcal{B}(T)\mu_0(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi) = 1$ from the mass conservation property of the linear Boltzmann equation and $\mu_T(\mathbb{R}^d_x \times \mathbb{R}^d_\xi) \leq 1$ from the properties of semiclassical measures. So, necessarily,

$$\mu_T\big(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi\big) = 1\,, \qquad \mu_T\big(\mathbb{R}^d_x \times \{0\}_\xi\big) = 0$$

and $\mu_T = \mathcal{B}(T)\mu_0$. Hence the result. □

Remark 2.3. Step 2 is the technical part and requires various estimates developed in this article.

Remark 2.4. Let us justify the scaling in the Weyl quantization. Physically the parameter h is the quotient of the microscopic scale over the macroscopic scale, either in time or in position. Thus if we consider an observable $b(X, \Xi)$ varying on a macroscopic scale, the corresponding observable on the microscopic scale will be $b(hx, \xi)$.

The scaling of the random field according to the covariance $hG(x - x')$ is done on a mesoscopic scale imposed by the kinetic regime. In microscopic variables, consider a particle moving among obstacles with a velocity $v \propto 1$ and a distance of interaction $R \propto 1$. During a time T the particle sweeps a volume of order vTR^{d-1}. In the kinetic regime it is assumed that during a long microscopic time $T = t/h$ with $t \propto 1$ the macroscopic time, the average particle encounters a number $\propto 1$ of obstacles. We denote by ρ the density of obstacles and thus obtain $\rho = 1/vTR^{d-1} \propto h$. To get this density of obstacles we need the distance between two nearest obstacles to be of order $h^{-1/d}$.

Thus we consider a Schrödinger equation of the form

$$i\partial_T \psi = -\Delta_x \psi + \mathcal{V}_\omega^h(x)\, \psi\,,$$

that is,

$$ih\partial_t \psi = -\Delta_x \psi + \mathcal{V}_\omega^h(x)\, \psi\,.$$

A translation invariant Gaussian random field of covariance $G(x - x')$, $\hat{G} = |\hat{V}|^2$, is of the form $V * W_\omega$, where W_ω is the spatial white noise and V describes the interaction potential. In the kinetic regime the obstacles are spread at the mesoscopic scale $h^{1/d}$. Only the white noise W_ω^h is rescaled (and not V) according to

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d; \mathbb{R})\,, \qquad \int \varphi(h^{1/d}x)\, W_\omega^h(x)\, \mathrm{d}x = \int \varphi(x)\, W_\omega(x)\, \mathrm{d}x\,,$$

i.e., $W_\omega^h(x) = hW_\omega(h^{1/d}x)$. Thus we get $\mathcal{V}_\omega^h = hV * W_\omega(h^{1/d}\cdot)$ and $G^h = hG$.

To prove Theorem 2.2 we first consider the case without the renewal of the stochastics, *i.e.*, $N = 1$ for short times in Sections 5, 6, 7 and then glue together the estimates obtained this way N times for N "big" in Section 8. To simplify the problem of finding estimates for short times we approximate the equation by a simpler one which is solved and studied in Section 5. In Section 6, using the solution to the approximated equation, we carry out explicit computations which give rise to the different terms of the dual linear Boltzmann equation. Then we control the error between the solutions of the approximated equation and the exact equation in Section 7. All these computations are done within the framework of quantum field theory. This allows us

- to use conveniently the geometric content of coherent states,
- to keep track of the different orders of importance of the different terms by using the Wick quantization with a parameter ε.

We expose the correspondence between the stochastic and Fock space viewpoints in Section 4.

Remark 2.5. Our initial data $(\rho^h)_{h\in(0,h_0]}$ are assumed to belong to $\mathcal{L}_1^+ L_x^2$ with $\mathrm{Tr}\, \rho^h = 1$. We thus make estimates for states ρ in $\mathcal{L}_1^+ L_x^2$, with $\mathrm{Tr}\, \rho = 1$ with constants independent of ρ.

3. The linear Boltzmann equation

Information on the linear Boltzmann equation can be found in the books of Dautray and Lions [12, 13] or Reed and Simon [34].

In Section 3 the set of values of functions is $\mathbb{R}$ when nothing is precised. We assume that $\sigma \in \mathcal{C}^\infty(\mathbb{R}^d_\xi \times \mathbb{R}^d_{\xi'})$ and $\sigma \geq 0$.

3.1. Formal definition.

Since all the objects we use are diagonal in $|\xi|$, the following notations are convenient.

Notation: Let $0 < r < r' < +\infty$, we define the Sobolev spaces

$$H^n[r, r'] = H^n(\mathbb{R}^d_x \times A_\xi[r, r'])$$

where $A_\xi[r, r']$ is the annulus $\{\xi \in \mathbb{R}^d,\, |\xi| \in (r, r')\}$ in the variable ξ. When there is no ambiguity we write A_ξ for $A_\xi[r, r']$. We also write $L^2[r, r']$ for $H^0[r, r']$.

Definition 3.1. The *linear Boltzmann equation* is formally the equation, with initial condition $\mu_{t=0} = \mu_0$,

$$\partial_t \mu = \{\mu, |\xi|^2\} + Q\mu\,,$$

where the *collision operator* Q is defined for $b \in L^2[r, r']$ by

$$Qb = Q_+ b - Q_- b\,, \tag{3.1}$$

with

$$Q_+ b(x, \xi) = \int_{\mathbb{R}^d_{\xi'}} b(x, \xi')\, \sigma(\xi, \xi')\, \delta\big(\, |\xi|^2 - |\xi'|^2 \,\big)\, \mathrm{d}\xi'\,,$$

$$Q_- b(x, \xi) = b(x, \xi) \int_{\mathbb{R}^d_{\xi'}} \sigma(\xi, \xi')\, \delta\big(\, |\xi|^2 - |\xi'|^2 \,\big)\, \mathrm{d}\xi'\,.$$

The dual linear Boltzmann equation with initial condition $b_{t=0} = b_0$ is

$$\partial_t b = -\{b, |\xi|^2\} + Qb = 2\xi.\partial_x b + Qb\,.$$

Remark 3.2. For a given ξ the integrals in the collision operator only involve the values of $\sigma(\xi, |\xi|\,\omega)$ and $b(x, |\xi|\,\omega)$ for $\omega \in \mathbb{S}^{d-1}$.

We show in Section 3.2 that the dual linear Boltzmann equation is solved by a group $(\mathcal{B}^T(t))_{t\in\mathbb{R}}$ of operators on $\mathcal{C}^0_\infty(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$ and in Section 3.3 that it defines by duality a group $(\mathcal{B}(t))_{t\in\mathbb{R}}$ of operators on $\mathcal{M}_b(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$.

3.2. Properties.

We recall here the main properties of the dual linear Boltzmann equation. (The arguments are the same as for the linear Boltzmann equation.)

We begin by solving the dual linear Boltzmann equation in $L^2[r, r']$ in the sense of semigroups.

Proposition 3.3. *Let $0 < r < r' < +\infty$.*

The operator

- *$2\xi.\partial_x$ generates a strongly continuous contraction semigroup on $L^2[r, r']$.*
- *Q is well defined and bounded on $H^n[r, r']$, with*

$$\|Q\|_{\mathcal{L}(H^n[r,r'])} \leq C_d \sup_{|\alpha|\leq n} \|\partial^\alpha \sigma\|_{\infty, A^2_\xi[r,r']}\,.$$

The group of space-translations $(e^{2t\xi.\partial_x})_t$ preserves $H^n[r, r']$.

- $2\xi.\partial_x + Q$ *generates a semigroup* $(\mathcal{B}^T(t))_{t\geq 0}$ *bounded by* $\exp(t\,\|Q\|_{\mathcal{L}(L^2[r,r'])})$ *since* Q *is bounded on* $L^2[r,r']$.

The strongly continuous group $(\mathcal{B}^T(t))_{t\geq 0}$ *preserves*

(1) *the Sobolev spaces* $H^n[r,r']$, *for* $n\in\mathbb{N}$,
(2) *the set of functions with compact support,*
(3) *the set of infinitely differentiable functions with compact support in* $\mathbb{R}^d_x \times A_\xi[r,r']$, $\mathcal{C}_0^\infty(\mathbb{R}^d_x \times A_\xi[r,r'])$,
(4) *the set of non-negative functions, for* $t\geq 0$.

Proof. The properties of generation of groups are clear.

Point (1) is a consequence of Proposition 3.3.

Point (2) follows from the Trotter approximation

$$\mathcal{B}^T(t) = \lim_{n\to\infty}\left(e^{2\frac{t}{n}\xi.\partial_x}e^{\frac{t}{n}Q}\right)^n,$$

the fact that Q is “local” in $(x,|\xi|)$, and that the speed of propagation of the space-translations is finite when $\xi\in A_\xi[r,r']$.

Point (3) follows from (1), (2) and

$$\mathcal{C}_0^\infty(\mathbb{R}^d_x \times A_\xi[r,r']) = \bigcap_{n=0}^{\infty} H^n\left[r,r'\right]\bigcap\{f,\,\mathrm{Supp}\,f\,\mathrm{compact}\}\ .$$

Point (4) follows from both the Trotter approximation

$$\mathcal{B}^T(t) = \lim_{n\to\infty}\left(e^{2\frac{t}{n}\xi.\partial_x}e^{\frac{t}{n}Q_+}e^{-\frac{t}{n}Q_-}\right)^n$$

and the fact that $e^{2\frac{t}{n}\xi.\partial_x}$ preserves the non-negative functions as a translation, $e^{\frac{t}{n}Q_+}$ preserves the non-negative functions for $t\geq 0$ because Q_+ does, $e^{-\frac{t}{n}Q_-}$ preserves the non-negative functions as a multiplication operator by a positive function. □

Since $\mathcal{C}_0^\infty(\mathbb{R}^d_x\times A_\xi)\subset D(2\xi.\partial_x)$ we can give the following result.

Proposition 3.4. *For all* $b_0\in\mathcal{C}_0^\infty(\mathbb{R}^d_x\times A_\xi)$, $b_t=\mathcal{B}^T(t)b_0$ *is the unique solution in* $\mathcal{C}^1(\mathbb{R}^+;L^2[r,r'])\cap\mathcal{C}^0(\mathbb{R}^+;D(2\xi.\partial_x))$ *to the Dual linear Boltzmann equation such that* $b_{t=0}=b_0$. *Moreover* $\forall t\in\mathbb{R}$, $b_t\in\mathcal{C}_0^\infty(\mathbb{R}^d_x\times A_\xi)$. *If* b_0 *is non-negative, then* $\forall t\geq 0$, b_t *is non-negative.*

3.3. **The linear Boltzmann equation.** The continuous functions vanishing at infinity and the Radon measures on a locally compact, Hausdorff space X are denoted by

$$\mathcal{C}^0_\infty(X) = \{f\in\mathcal{C}^0(X)\,,\,\forall\varepsilon>0\,,\,\exists K\,\text{compact s.t.}\,\forall x\notin K\,,\,|f(x)|<\varepsilon\}\,,$$
$$\mathcal{M}_b(X) = (\mathcal{C}^0_\infty(X))'\,.$$

Proposition 3.5. *The semigroup* $(\mathcal{B}^T(t))_{t\geq 0}$ *defined on* $\mathcal{C}_0^\infty(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)$ *extends to a strongly continuous group on* $(\mathcal{C}^0_\infty(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi),\|\cdot\|_\infty)$ *and defines by duality a (weak* continuous) group* $\mathcal{B}(t)$ *on* $\mathcal{M}_b(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)$.

Proof. Using a partition of the unity, $\mathcal{B}^T(t)$ extends to $\mathcal{C}^\infty(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)$. Since $\mathcal{B}^T(t)$ is positive, we have $\mathcal{B}^T(t)(\|b\|_\infty\pm b)\geq 0$ for all b in $\mathcal{C}_0^\infty(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)$ and so $\|\mathcal{B}^T(t)\,b\|_\infty\leq\|b\|_\infty$. The group $\mathcal{B}^T(t)$ thus extends continuously to $\mathcal{C}^0_\infty(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)$. □

Definition 3.6. The *linear Boltzmann group* $(\mathcal{B}(t))$ is defined on $\mathcal{M}_b(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$ by duality: let $\mu \in \mathcal{M}_b(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$, then, for any $t \in \mathbb{R}$,

$$\forall b \in \mathcal{C}^0_\infty(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi), \qquad \langle \mathcal{B}(t)\mu, b\rangle = \langle \mu, \mathcal{B}^T(t) b\rangle \,.$$

3.4. **A Trotter-type approximation.** This Section provides a result in the spirit of Trotter's approximation $(e^{A/N}e^{B/N})^N \to e^{A+B}$ useful to deal with the renewal of the stochasticity.

Proposition 3.7. *Let $b \in \mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$, $T > 0$ and $n \in \mathbb{N}$. There are constants $C_{n,Q}$ and $C_{T,b}$ such that for all $N \in \mathbb{N}^*$*

$$\mathcal{N}_n\big(e^{T(2\xi.\partial_x + Q)} b - \big(e^{\frac{T}{N}Q} e^{\frac{T}{N}2\xi.\partial_x}\big)^N b\big) \le e^{T(2n + C_{n,Q})} C_{T,b} \frac{T^2}{N} \,.$$

where, for $n \in \mathbb{N}$, $\mathcal{N}_n(b) := \sup_{|\alpha| \le n} \|\partial^\alpha b\|_\infty$.

Notation 3.8. Let $Q_t = e^{t2\xi.\partial_x} Q e^{-t2\xi.\partial_x} \in \mathcal{L}(L^2[r,r'])$, *i.e.* $Q_t = Q_{+,t} - Q_-$ with

$$Q_{+,t} b(x,\xi) = \int_{\mathbb{R}^d_{\xi'}} \sigma(\xi,\xi')\,\delta\big(|\xi|^2 - |\xi'|^2\big)\, b(x - 2t(\xi' - \xi), \xi')\,\mathrm{d}\xi' \,.$$

Let also $Q_{-,t} = Q_-$ to have consistent notations in the sequel.

Let $G_Q(t,t_0)$ be the dynamical system associated with the one parameter family (Q_t) in $\mathcal{C}(\mathbb{R}; \mathcal{L}(L^2[r,r']))$ given by

$$\begin{cases} \partial_t b_t = Q_t\, b_t \\ b_{t=t_0} = b_0 \in L^2_{x,\xi} \end{cases}, \qquad b_t = G_Q\,(t,t_0)\, b_0 \,.$$

Note the relation $\mathcal{B}^T(t) = G_Q(t,0) e^{2t\xi.\partial_x} = e^{2t\xi.\partial_x} G_Q(0,-t)$.

For $b \in \mathcal{C}^\infty_0(\mathbb{R}^d_x \times A_\xi[r,r'])$, let

$$\mathcal{N}_n(Q) = \sup_{b \ne 0} \frac{\mathcal{N}_n(Qb)}{\mathcal{N}_n b} \text{ and } \mathcal{N}_{n+1,n}(s, Q - Q_s) = \sup_{b \ne 0} \frac{\mathcal{N}_n\big((Q - Q_s)b\big)}{s\,(1 + 2\,|s|)^n\, \mathcal{N}_{n+1} b} \,.$$

Lemma 3.9. *For any $n \in \mathbb{N}$, $s \ge 0$ and $b \in \mathcal{C}^\infty_0(\mathbb{R}^d_x \times A_\xi[r,r'])$, there exist constants C_1, and C_2 depending on d, r and r' such that*

(1) $\mathcal{N}_n(Q) \le C_1$,
(2) $\mathcal{N}_{n+1,n}(t, Q - Q_t) \le C_2$,
(3) $\mathcal{N}_n(e^{2t\xi.\partial_x} b) \le (1 + 2\,|t|)^n\, \mathcal{N}_n(b)$.

Proof. The first point is clear from the integral expression of Qb.

For the second point differentiate and estimate the integral formula for $b\,(x - 2t\xi, \xi) - b\,(x,\xi)$, with $|\alpha| \le n$,

$$\begin{aligned} \big|\partial^\alpha\big(b(x - 2t\xi, \xi) - b(x,\xi)\big)\big| &\le \int_0^t \big|\partial^\alpha\big(2\xi.\partial_x b(x - 2s\xi, \xi)\big)\big|\,\mathrm{d}s \\ &\le 2\,|\xi|\, t\,(1 + 2t)^n\, \mathcal{N}_{n+1}(b) \,. \end{aligned}$$

The last point results from $\big(e^{2t\xi.\partial_x} b\big)(x,\xi) = b(x + 2t\xi, \xi)$. □

Lemma 3.10. *Let b, $\tilde{b} \in \mathcal{C}^\infty_0\left(\mathbb{R}^d_x \times A_\xi\,[r,r']\right)$, then for all $t \ge 0$,*

$$e^{tQ}\tilde{b} - G_Q(t,0) b = e^{tQ}(\tilde{b} - b) + \int_0^t e^{(t-s)Q}(Q - Q_s) G_Q(s,0) b\,\mathrm{d}s$$

and we have the estimate

$$\mathcal{N}_n\big(e^{tQ}\tilde{b} - G_Q(t,0)b\big) \leq e^{t\mathcal{N}_n Q}\,\mathcal{N}_n(\tilde{b}-b) \\ + t^2(1+2t)^n e^{t\mathcal{N}_n Q} \sup_{s\in[0,t]} \big\{\mathcal{N}_{n+1,n}(s, Q-Q_s)\,\mathcal{N}_{n+1}\big(G_Q(s,0)\big)\big\}\,\mathcal{N}_{n+1}(b)\,.$$

Proof. The equality is clear once we have computed that both sides satisfy the equation

$$\partial_t \Delta_t = Q\Delta_t + (Q-Q_t)G_Q(t,0)b\,.$$

The inequality then follows from Lemma 3.9. □

Proof of Proposition 3.7. We fix N and forget the N's in the notations concerning $\tilde{b}$. We set $b_t = \mathcal{B}^T(t)\,b$ and define $\tilde{b}_t$ piecewise on $[0,T]$ by setting $t_k = \frac{kT}{N}$, $\tilde{b}_{t_k} = \big(e^{\frac{T}{N}Q}e^{\frac{T}{N}2\xi.\partial_x}\big)^k b_0$ and, for $t\in[t_k,t_{k+1})$, $\tilde{b}_t = e^{(t-t_k)Q}e^{(t-t_k)2\xi.\partial_x}\tilde{b}_{t_k}$. Let $\delta_k = \mathcal{N}_n\big(b_{t_k}-\tilde{b}_{t_k}\big)$; we get

$$e^{\frac{T}{N}Q}e^{\frac{T}{N}2\xi.\partial_x}\tilde{b}_{t_k} - e^{\frac{T}{N}(2\xi.\partial_x+Q)}b_{t_k} = e^{\frac{T}{N}Q}e^{\frac{T}{N}2\xi.\partial_x}\tilde{b}_{t_k} - G_Q\big(\tfrac{T}{N},0\big)e^{\frac{T}{N}2\xi.\partial_x}b_{t_k}$$

and we can then use Lemma 3.10 to obtain

$$\begin{aligned}\delta_{k+1} &\leq e^{\frac{T}{N}\mathcal{N}_n Q}\left(1+2\tfrac{T}{N}\right)^n\delta_k + \left(\tfrac{T}{N}\right)^2\left(1+2\tfrac{T}{N}\right)^n e^{\frac{T}{N}\mathcal{N}_n Q} \\ &\qquad \sup_{s\in[t_k,t_{k+1}]}\mathcal{N}_{n+1,n}\big(s-t_k, Q-Q_{s-t_k}\big) \\ &\qquad \sup_{s\in[t_k,t_{k+1}]}\mathcal{N}_{n+1}\big(G_Q(s-t_k,0)\,e^{\frac{T}{N}2\xi.\partial_x}b_{t_k}\big) \\ &\leq e^{\frac{T}{N}\mathcal{N}_n Q}e^{2\frac{nT}{N}}\delta_k + \left(\tfrac{T}{N}\right)^2 e^{\frac{T}{N}\mathcal{N}_n Q}C_{N,T}\end{aligned}$$

where we introduced

$$C_{N,T,b} = \big(1+2\tfrac{T}{N}\big)^n \sup_{s\in[0,T/N]}\mathcal{N}_{n+1,n}(s,Q-Q_s) \\ \sup_{k\in\{0,\dots,N-1\}}\sup_{s\in[0,T/N]}\mathcal{N}_{n+1}\big(G_Q(s,0)\,e^{-\frac{T}{N}Q}b_{t_{k+1}}\big)\,.$$

Then we get the recursive formula

$$\delta_{k+1} \leq e^{\frac{T}{N}(2n+\mathcal{N}_n Q)}\,\delta_k + C_{N,T,b}\left(\tfrac{T}{N}\right)^2 e^{\frac{T}{N}\mathcal{N}_n Q}$$

so that

$$\delta_N \leq e^{T\,(2n+\mathcal{N}_n Q)}\,C_{N,T,b}\frac{T^2}{N}\,.$$

The only thing remaining is to observe that $C_{N,T,b}\leq C_{T,b}$, with

$$C_{T,b} := (1+2T)^n \sup_{s\in[0,T]}\mathcal{N}_{n+1,n}(s,Q-Q_s)\sup_{s_j\in[0,T]}\mathcal{N}_{n+1}\big(G_Q(s_1,0)\,e^{-s_2Q}b_{s_3}\big)$$

and for a fixed T this quantity $C_{T,b}$ is finite, so that we get the result. □

4. From stochastics to the Fock space

4.1. **The second quantization.** The method of second quantization is exposed in the books of Berezin [7] and Bratteli and Robinson [9], an introduction to quantum field theory and second quantization can be found in the book of Folland [17]. The series of articles of Ginibre and Velo [23, 24, 25, 26] uses this framework with a small parameter to handle classical or mean field limits by extending the Hepp method [28]. We use the notation and framework of articles of Ammari and Nier [1, 2] to handle the second quantization with a small parameter. For the convenience of the reader we expose briefly this framework.

Most of the operators on the Fock space in this article arise as Wick quantizations of polynomials.

Definition 4.1. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex separable Hilbert space (the scalar product is $\mathbb{C}$-antilinear with respect to the left variable). The symmetric tensor product is denoted by $\vee$. The *polynomials* with variable in $\mathcal{H}$ are the finite linear combinations of monomials $Q : \mathcal{H} \to \mathbb{C}$ of the form

$$Q\,(z) = \langle z^{\vee q}, \tilde{Q} z^{\vee p}\rangle$$

where $p, q \in \mathbb{N}$, $\tilde{Q} \in \mathcal{L}(\mathcal{H}^{\vee p}, \mathcal{H}^{\vee q})$ and $\langle\cdot,\cdot\rangle$ denotes the scalar product on $\mathcal{H}^{\vee q}$. The set of such polynomials is denoted by $\mathcal{P}(\mathcal{H})$.

The symmetric *Fock space* associated to $\mathcal{H}$ is

$$\Gamma\mathcal{H} = \bigoplus_{n=0}^{\infty} \Gamma_n \mathcal{H}$$

with $\Gamma_n\mathcal{H} = \mathcal{H}^{\vee n}$ the Hilbert completed n-th symmetric power of $\mathcal{H}$ and the sum is completed, the set of *finite particle vectors* $\Gamma_F\mathcal{H}$ is defined as the Fock space but with an algebraic sum.

Let $\varepsilon > 0$. The *Wick quantization* of a polynomial is defined as the linear combination of the Wick quantizations of its monomials, and for a monomial Q we define $Q^{Wick} : \Gamma_F\mathcal{H} \to \Gamma_F\mathcal{H}$ as the linear operator which vanishes on $\mathcal{H}^{\vee n}$ for $n < p$ and for $n \geq 0$

$$Q^{Wick}\big|_{\mathcal{H}^{\vee n+p}} = \frac{\sqrt{(n+p)!(n+q)!}}{n!}\,\varepsilon^{\frac{p+q}{2}}\,(\tilde{Q} \vee \mathrm{Id}_{\mathcal{H}^{\vee n}}) \in \mathcal{L}\big(\mathcal{H}^{\vee n+p}, \mathcal{H}^{\vee n+q}\big)\,.$$

The field operator $\Phi_\varepsilon(f)$ ($f \in \mathcal{H}$) is the closure of the essentially self-adjoint operator $(\langle z, f\rangle + \langle f, z\rangle)^{Wick}/\sqrt{2}$. Using the Weyl operator $W(f) = \exp(i\Phi_\varepsilon(f))$ the coherent state $E(f) = W\big(\frac{\sqrt{2}}{i\varepsilon} f\big)\,\Omega$ can be defined, where $\Omega = (1, 0, 0, \dots) \in \Gamma\mathcal{H}$ is the empty state. The Weyl operators satisfy the relation

$$W(f)\,W(g) = e^{-\frac{i\varepsilon}{2}\Im\langle f,g\rangle} W(f+g)\,.$$

The second quantization $\mathrm{d}\Gamma_\varepsilon(A)$ of a self-adjoint operator A on $\mathcal{H}$ is

$$\mathrm{d}\Gamma_\varepsilon(A)\big|_{D(A)^{\vee n,\mathrm{alg}}} = \varepsilon\,(A \otimes \mathrm{Id}_{\mathcal{H}} \otimes \dots \otimes \mathrm{Id}_{\mathcal{H}} + \dots + \mathrm{Id}_{\mathcal{H}} \otimes \dots \otimes \mathrm{Id}_{\mathcal{H}} \otimes A)$$

and for a unitary U on $\mathcal{H}$, the unitary operator $\Gamma(U)$ on $\Gamma\mathcal{H}$ is defined by

$$\Gamma(U)|_{\mathcal{H}^{\vee n}} = U^{\vee n} = U \otimes \dots \otimes U$$

and thus $\Gamma(e^{itA}) = \exp\big(\frac{it}{\varepsilon}\,\mathrm{d}\Gamma_\varepsilon(A)\big)$.

4.2. **The expression of the dynamic in the Fock space.** The relation between Gaussian random processes and the Fock space is treated in the books of Simon [35] and Glimm and Jaffe [27], we recall a theorem about this relation.

Theorem 4.2. *Let $\mathcal{V}^h(x)$ be the centered, translation invariant, gaussian random field with covariance $hG(x-y)$ such that $\hat{G} = |\hat{V}|^2$ for some $V \in \mathcal{S}(\mathbb{R}^d;\mathbb{R})$. The symmetric Fock space $\Gamma L^2(\mathbb{R}^d;\mathbb{C})$ is unitarily equivalent to $L^2(\Omega_\mathbb{P}, \mathbb{P};\mathbb{C})$ under a unitary $D : \Gamma\mathcal{H}_\mathbb{C} \to L^2(\Omega_\mathbb{P}, \mathbb{P};\mathbb{C})$ such that*

- *$D\,\Omega = 1$,*

- $D\sqrt{2h}\Phi_1(\tau_x V)\,D^{-1} = \mathcal{V}^h(x)$, *with* $\mathcal{V}^h(x)$ *seen as a multiplication operator on* $L^2(\Omega_{\mathbb{P}}, \mathbb{P}; \mathbb{C})$.

For Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, $\mathrm{Tr}_{\mathcal{H}'}[A]$ denotes the partial trace of an operator $A \in \mathcal{L}_1(\mathcal{H} \otimes \mathcal{H}')$, $\mathrm{Tr}_{\mathcal{H}}[\mathrm{Tr}_{\mathcal{H}'}[A]B] = \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}'}[A(B \otimes I_{\mathcal{H}'})]$, $\forall B \in$ on $\mathcal{H} \otimes \mathcal{H}'$.

Proposition 4.3. *Let* $H_h = -\Delta_x + \sqrt{2h}\Phi_1(\tau_x V)$, *with* $\tau_x f(y) = f(y-x)$ *for* $x \in \mathbb{R}^d$ *and* $f \in L^2_y$. *Then*

$$\mathcal{G}^h_t(\rho) = \mathrm{Tr}_{\,\Gamma L^2_y}\left[e^{-i\frac{t}{h}H_h}\,\rho \otimes |\Omega\rangle\langle\Omega|\,e^{i\frac{t}{h}H_h}\right].$$

Proof. In the stochastic presentation we can express the integral in ω in the definition of $\mathcal{G}^h_t$ as a partial trace

$$\begin{aligned}\mathcal{G}^h_t(\rho) &= \int e^{-i\frac{t}{h}H_{h,\omega}}\rho 1(\omega)1(\omega)e^{i\frac{t}{h}H_{h,\omega}}\,\mathrm{d}\mathbb{P}(\omega)\,.\\ &= \mathrm{Tr}_{\,L^2(\Omega_{\mathbb{P}},\mathbb{P})}\left[\int^{\oplus} e^{-i\frac{t}{h}H_{h,\omega}}\,\mathrm{d}\mathbb{P}(\omega)\rho \otimes |1\rangle\langle 1| \int^{\oplus} e^{i\frac{t}{h}H_{h,\omega'}}\,\mathrm{d}\mathbb{P}(\omega')\right].\end{aligned}$$

Using the isomorphism $\mathcal{U} := \mathrm{Id}_{L^2_x} \otimes D : L^2_x \otimes \Gamma L^2_y \to L^2_x \otimes L^2(\Omega_{\mathbb{P}}, \mathbb{P})$ we get

$$\mathcal{U}^* \int^{\oplus} e^{-i\frac{\Delta t}{h}H_{h,\omega}}\,\mathrm{d}\mathbb{P}(\omega)\,\mathcal{U} = e^{-i\frac{\Delta t}{h}H_h}\,, \quad \text{and} \quad \mathcal{U}^*\,\rho \otimes |1\rangle\,\langle 1|\,\mathcal{U} = \rho \otimes |\Omega\rangle\,\langle\Omega|$$

with $H_h := \mathcal{U}^*\left(\int^{\oplus} H_{h,\omega}\,\mathrm{d}\mathbb{P}(\omega)\right)\mathcal{U} = -\Delta_x + \sqrt{2h}\Phi_1(\tau_x V)$. Hence the result. □

4.3. **Existence of the dynamic.** We show that the dynamic of the system is well defined. Since we work with a fixed $h > 0$ the value of h is here irrelevant and we set $h = 1$ in this section to clarify our exposition. We write for short

- $-\Delta_x$ for the operator $-\Delta_x \otimes \mathrm{Id}_{\Gamma L^2_y}$,
- N for the operator $\mathrm{Id}_{L^2_x} \otimes N$ with $N = \mathrm{d}\Gamma_1(\mathrm{Id}_{L^2_y})$ the number operator on ΓL^2_y and
- $\Phi_1(\tau.V)$ the operator on $L^2(\mathbb{R}^d; \Gamma L^2_y) \simeq L^2(\mathbb{R}^d) \otimes \Gamma L^2_y$ defined by $u \mapsto \Phi_1(\tau.V)u$ with $[\Phi_1(\tau.V)u](x) := [\Phi_1(\tau_x V)][u(x)]$.

Proposition 4.4. *If* V *belongs to the Sobolev space* $H^2(\mathbb{R}^d)$, *then*

$$H = -\Delta_x + \sqrt{2}\Phi_1(\tau.V)\,,$$

is essentially self-adjoint on $D' := \mathcal{C}^\infty_0(\mathbb{R}^d) \otimes^{alg} \Gamma_F L^2_y$ *and its closure is essentially self-adjoint on any other core for* $N' = \mathrm{Id} - \Delta_x + N$.

Proof. We still denote by N' the closure of the essentially self-adjoint operator N' defined on D'. Then D' is a core for this operator. We remark that $N' \geq I$ on D' and thus also on $D(N')$ as D' is a core for N'.

We verify the two estimates needed for Nelson's commutator theorem (see the book of Reed and Simon [33]). Let $u \in D'$, then

$$\begin{aligned}\|Hu\|_{L^2_x\otimes\Gamma L^2_y} &\leq \|-\Delta_x u\|_{L^2_x\otimes\Gamma L^2_y} + 2\,\|V\|_{L^2}\left\|\sqrt{N+1}u\right\|_{L^2_x\otimes\Gamma L^2_y}\,,\\ &\leq (1 + 2\,\|V\|_{L^2})\,\|N'u\|_{L^2_x\otimes\Gamma L^2_y}\,.\end{aligned}$$

In the sense of quadratic forms

$$\begin{aligned}[H, N'] &= \sqrt{2}[\Phi(\tau.V), -\Delta_x + N]\\ &= \sqrt{2}\Phi(\tau.\nabla V).\nabla_x + \sqrt{2}\Phi(\tau.\Delta V) + (a^*(\tau.V) - a(\tau.V))\end{aligned}$$

so that $|\langle Hu, N'u\rangle - \langle N'u, Hu\rangle| \leq 6\|V\|_{H^2}\|N'^{1/2}u\|^2$ which achieves the proof. □

4.4. **The scaling for field operators.** The ε parameter is an intermediate scale which allows to easily identify the graduation in Wick powers. We set $\mathrm{Ad}\{A\}[B] = ABA^{-1}$. Let $(D_\varepsilon f)(y) = \varepsilon^{-d/2} f\left(\frac{y}{\varepsilon}\right)$ and

$$H_{h,\varepsilon} = \mathrm{Ad}\left\{\mathrm{Id}_{L^2_x} \otimes \Gamma D_\varepsilon\right\}[H_h] = -\Delta_x \otimes I_{\Gamma_y} + \sqrt{2h}\Phi\big(\varepsilon^{-d/2} V(x - \tfrac{y}{\varepsilon})\big)\,.$$

5. An approximated equation and its solution

5.1. **Space translation in the fields and Fourier transform.**

Notation 5.1. For an object $X = (X_1, \ldots, X_d)$ with d components, like $\xi \in \mathbb{R}^d$, $D_x = (\partial_{x_1}, \ldots, \partial_{x_d})$ or $\mathrm{d}\Gamma_\varepsilon(D_y)$, let $X^{\cdot 2} := X_1^2 + \cdots + X_d^2$.

We want to work with a field operator with no dependence in x. Then we recall that the translation τ_x of x can be written as $e^{-ix.D_y}$ and thus

$$\Gamma\big(e^{i\varepsilon x.D_y}\big)\, H_{h,\varepsilon}\, \Gamma\big(e^{-i\varepsilon x.D_y}\big) = (\mathrm{d}\Gamma_\varepsilon(D_y) - D_x)^{\cdot 2} + \sqrt{2}\Phi_\varepsilon\Big(\varepsilon^{-d/2}\sqrt{\tfrac{h}{\varepsilon}} V\left(-\tfrac{y}{\varepsilon}\right)\Big)$$

where we use the ε-dependent operator $\mathrm{d}\Gamma_\varepsilon$. A conjugation by the Fourier transform in both the particle and the field variables yields a new expression for the Hamiltonian, and an approximated version

$$\hat{H}_{h,\varepsilon} = \xi^{\cdot 2} - \mathrm{d}\Gamma_\varepsilon(2\xi.\eta) + \mathrm{d}\Gamma_\varepsilon(\eta)^{\cdot 2} + \sqrt{2}\Phi_\varepsilon(f_{h,\varepsilon})\,,$$
$$\hat{H}^{app}_{h,\varepsilon} = \xi^{\cdot 2} + \mathrm{d}\Gamma_\varepsilon\left(\varepsilon\eta^{\cdot 2} - 2\xi.\eta\right) + \sqrt{2}\Phi_\varepsilon\left(f_{h,\varepsilon}\right)\,,$$

with $f_{h,\varepsilon}(\eta) = \varepsilon^{d/2}\sqrt{\frac{h}{\varepsilon}}\hat{V}(-\varepsilon\eta)$, *i.e.* $\hat{H}_{h,\varepsilon} = Q^{Wick}_{h,\varepsilon}$ and $\hat{H}^{app}_{h,\varepsilon} = Q^{app,Wick}_{h,\varepsilon}$ with

$$Q_{h,\varepsilon}(z) = \xi^{\cdot 2} + \langle z, (\varepsilon\eta^{\cdot 2} - 2\xi.\eta)z\rangle + \langle z, \eta z\rangle^{\cdot 2} + 2\Re\langle z, f_{h,\varepsilon}\rangle\,,$$
$$Q^{app}_{h,\varepsilon}(z) = \xi^{\cdot 2} + \langle z, (\varepsilon\eta^{\cdot 2} - 2\xi.\eta)z\rangle + 2\Re\langle z, f_{h,\varepsilon}\rangle\,.$$

Note that in the approximated Hamiltonian we neglect the quartic part $\langle z, \eta z\rangle^{\cdot 2}$. The evolution associated with the approximated Hamiltonian is explicitely solvable.

Definition 5.2. For $\rho \in \mathcal{L}_1\left(L^2_x\right)$, let

$$\boldsymbol{\rho}_t = \mathrm{Ad}\left\{e^{-i\frac{t}{\varepsilon}H_{h,\varepsilon}}\right\}[\rho \otimes |\Omega\rangle\langle\Omega|]\,, \qquad \boldsymbol{\rho}^{app}_t = \mathrm{Ad}\left\{e^{-i\frac{t}{\varepsilon}H^{app}_{h,\varepsilon}}\right\}[\rho \otimes |\Omega\rangle\langle\Omega|]\,,$$
$$\hat{\boldsymbol{\rho}}_t = \mathrm{Ad}\left\{e^{-i\frac{t}{\varepsilon}\hat{H}_{h,\varepsilon}}\right\}[\hat{\rho} \otimes |\Omega\rangle\langle\Omega|]\,, \qquad \hat{\boldsymbol{\rho}}^{app}_t = \mathrm{Ad}\left\{e^{-i\frac{t}{\varepsilon}\hat{H}^{app}_{h,\varepsilon}}\right\}[\hat{\rho} \otimes |\Omega\rangle\langle\Omega|]\,,$$
$$\rho^\varepsilon_t = \mathrm{Tr}_{\,\Gamma L^2_x}[\boldsymbol{\rho}_t]\,, \qquad \rho^{\varepsilon,app}_t = \mathrm{Tr}_{\,\Gamma L^2_x}[\boldsymbol{\rho}^{app}_t]\,.$$

This definition is consistent with the previous one given for ρ^h_t as $\rho^h_t = \rho^\varepsilon_{\frac{\varepsilon}{h}t}$ and the dilatation acts only in the Fock space part of $L^2_x \otimes \Gamma L^2_y$.

5.2. **The solution of the approximated equation.**

Proposition 5.3. *For $\psi_0 \in L^2_x$, let*

$$\hat{\Psi}_{h,\varepsilon,t} = e^{-i\frac{t}{\varepsilon}\hat{H}_{h,\varepsilon}}\Omega \otimes \hat{\psi}_0 \qquad \textit{and} \qquad \hat{\Psi}^{app}_{h,\varepsilon,t} = e^{-i\frac{t}{\varepsilon}\hat{H}^{app}_{h,\varepsilon}}\Omega \otimes \hat{\psi}_0\,, \tag{5.1}$$

$$z_{h,\varepsilon,t} = -i\int_0^t e^{-i\frac{s}{\varepsilon}(\varepsilon^2\eta^{\cdot 2} - 2\xi.\varepsilon\eta)} f_{h,\varepsilon}\,\mathrm{d}s \tag{5.2}$$

$$\omega_{h,\varepsilon,t} = t\xi^2 + \int_0^t \Re\langle z_s, f_{h,\varepsilon}\rangle\,\mathrm{d}s\,. \tag{5.3}$$

Then

(1) $\hat{\Psi}^{app}_{h,\varepsilon,t} = e^{-i\frac{\omega_{h,\varepsilon,t}}{\varepsilon}} W\big(\frac{\sqrt{2}}{i\varepsilon} z_{h,\varepsilon,t}\big)\, \Omega \otimes \hat{\psi}_0$.

(2) *There is a constant $C_{G,d}$ depending on G and the dimension d such that*

$$\left\| |\eta|^\nu \, z_{h,\varepsilon,t} \right\|_{L^2_\eta} \leq C_{G,d} \big(\tfrac{ht}{\varepsilon}\big)^{1/2} \varepsilon^{1/2-\nu} \,.$$

(3) *Let $T_0 > 0$. There is a constant $C_{T_0,G,d}$ such that for $\frac{ht}{\varepsilon} \leq T_0$,*

$$\left\| \hat{\Psi}_{h,\varepsilon,t} - \hat{\Psi}^{app}_{h,\varepsilon,t} \right\| \leq C_{T_0,G,d} \big(\tfrac{ht}{\varepsilon}/\sqrt{h}\big)^2 \,.$$

(4) *For both $\hat{\Psi}^\sharp_{h,\varepsilon,t} = \hat{\Psi}_{h,\varepsilon,t}$ and $\hat{\Psi}^\sharp_{h,\varepsilon,t} = \hat{\Psi}^{app}_{h,\varepsilon,t}$*

$$\left\| (\varepsilon + N_\varepsilon)^{1/2} \hat{\Psi}^\sharp_{h,\varepsilon,t} \right\| \leq C_d \Big(\sqrt{\varepsilon} + \sqrt{\tfrac{t}{2} \tfrac{ht}{\varepsilon} \|\hat{G}\|_{L^1}} \Big) \,.$$

First we get rid of the quadratic part $\mathrm{d}\Gamma_\varepsilon$. Let

- $\tilde{\hat{\Psi}}_{h,\varepsilon,t} = e^{i\frac{t}{\varepsilon}\xi^{\cdot 2}} e^{i\frac{t}{\varepsilon}\,\mathrm{d}\Gamma_\varepsilon(\varepsilon\eta^{\cdot 2} - 2\xi.\eta)} \hat{\Psi}_{h,\varepsilon,t}$ and $\tilde{\hat{\Psi}}^{app}_{h,\varepsilon,t} = e^{i\frac{t}{\varepsilon}\xi^{\cdot 2}} e^{i\frac{t}{\varepsilon}\,\mathrm{d}\Gamma_\varepsilon(\varepsilon\eta^{\cdot 2} - 2\xi.\eta)} \hat{\Psi}^{app}_{h,\varepsilon,t}$,
- $\tilde{f}_{h,\varepsilon,t} = e^{i\frac{t}{\varepsilon}(\varepsilon^2\eta^{\cdot 2} - 2\xi.\varepsilon\eta)} f_{h,\varepsilon}$,
- $\tilde{z}_{h,\varepsilon,t} = -i \int_0^t \tilde{f}_{h,\varepsilon,s}\, \mathrm{d}s$,
- $\tilde{\omega}_{h,\varepsilon,t} = \int_0^t \Re\langle \tilde{z}_{h,\varepsilon,s}, \tilde{f}_{h,\varepsilon,s} \rangle\, \mathrm{d}s$.

It is then enough to prove the results with the objects with a $\sim$ sign.

Lemma 5.4. *Then $\tilde{\hat{\Psi}}_t$ (resp. $\tilde{\hat{\Psi}}^{app}_t$) is solution of the equation*

$$i\varepsilon\partial_t \tilde{\hat{\Psi}}_{h,\varepsilon,t} = \tilde{Q}^{Wick}_{h,\varepsilon} \tilde{\hat{\Psi}}_{h,\varepsilon,t} \quad (\textit{resp.} \quad i\varepsilon\partial_t \tilde{\hat{\Psi}}^{app}_{h,\varepsilon,t} = \tilde{Q}^{app,Wick}_{h,\varepsilon} \tilde{\hat{\Psi}}^{app}_{h,\varepsilon,t}\,)$$

with initial condition $\Omega \otimes \hat{\psi}_0$, $\tilde{Q}_{h,\varepsilon,t}(z) = 2\Re\langle z, \tilde{f}_{h,\varepsilon,t} \rangle + \langle z, \eta z \rangle^{\cdot 2}$ (resp. $\tilde{Q}^{app}_{h,\varepsilon,t}(z) = 2\Re\langle z, \tilde{f}_{h,\varepsilon,t} \rangle$).

The function $\tilde{z}_{h,\varepsilon,t}$ is the solution of $i\partial_t \tilde{z}_{h,\varepsilon,t} = \partial_{\bar{z}} \tilde{Q}^{app}_{h,\varepsilon,t}(\tilde{z}_{h,\varepsilon,t}) = \tilde{f}_{h,\varepsilon,t}$, with initial condition $\tilde{z}_{h,\varepsilon,0} = 0$

Proof. Indeed

$$\begin{aligned} i\varepsilon\partial_t \tilde{\hat{\Psi}}_t &= i\varepsilon\partial_t [e^{i\frac{t}{\varepsilon}\xi^{\cdot 2}} e^{i\frac{t}{\varepsilon}\,\mathrm{d}\Gamma_\varepsilon(\varepsilon\eta^{\cdot 2} - 2\xi.\eta)} \hat{\Psi}_t] \\ &= e^{i\frac{t}{\varepsilon}\xi^{\cdot 2}} e^{i\frac{t}{\varepsilon}\,\mathrm{d}\Gamma_\varepsilon(\varepsilon\eta^{\cdot 2} - 2\xi.\eta)} [2\Re\langle z, f \rangle + \langle z, \eta z \rangle^{\cdot 2}]^{Wick} \hat{\Psi}_t \\ &= [2\Re\langle z, e^{it(\varepsilon\eta^{\cdot 2} - 2\xi.\eta)} f \rangle + \langle z, \eta z \rangle^{\cdot 2}]^{Wick} e^{i\frac{t}{\varepsilon}\xi^{\cdot 2}} e^{i\frac{t}{\varepsilon} d\Gamma_\varepsilon(\varepsilon\eta^{\cdot 2} - 2\xi.\eta)} \hat{\Psi}_t \\ &= \tilde{Q}^{Wick}_t \tilde{\hat{\Psi}}_t \,. \end{aligned}$$

And we can proceed analogously with $\tilde{\Psi}^{app}_t$. $\square$

Proof of Proposition 5.3. Point (1) follows from applying $i\varepsilon\partial_t$ to the right hand side:

$$\begin{aligned} & i\varepsilon\partial_t e^{-i\frac{\tilde{\omega}_t}{\varepsilon}} W\big(\tfrac{\sqrt{2}}{i\varepsilon}\tilde{z}_t\big)\, \Omega \otimes \hat{\psi}_0 \\ &= \Big(\partial_t\tilde{\omega} - i\varepsilon \frac{i\varepsilon}{2} \Im\big\langle \tfrac{\sqrt{2}}{i\varepsilon}\tilde{z}_t, -\tfrac{\sqrt{2}}{\varepsilon}\tilde{f}_t \big\rangle + i\varepsilon i \Phi_\varepsilon\big(-\tfrac{\sqrt{2}}{\varepsilon}\tilde{f}_t\big) \Big) e^{-i\frac{\omega_t}{\varepsilon}} W\big(\tfrac{\sqrt{2}}{i\varepsilon}\tilde{z}_t\big)\, \Omega \otimes \hat{\psi}_0 \\ &= \Big(\partial_t\tilde{\omega} - \Im\big\langle \tfrac{1}{i}\tilde{z}_t, \tilde{f}_t \big\rangle + \sqrt{2}\Phi_\varepsilon(\tilde{f}_t) \Big) \tilde{\hat{\Psi}}^{app}_t \end{aligned}$$

since $\frac{1}{t}\langle \varphi, [W(z + tu) - W(z)]\psi \rangle \xrightarrow[t\to 0]{} \langle \varphi, [-\frac{i\varepsilon}{2}\Im\langle z, u \rangle + i\Phi_\varepsilon(u)] W(z)\psi \rangle$.

For Point (2) we compute

$$\left\||\eta|^{\nu}\tilde{z}_{h,\varepsilon,t}\right\|^2_{L^2_\eta} = \int_0^t\int_0^t\int_{\mathbb{R}^d_\eta} e^{i\frac{s-s'}{\varepsilon}(\varepsilon^2\eta^{\cdot 2}-2\xi.\varepsilon\eta)}|\eta|^{2\nu}|f_{h,\varepsilon}(\eta)|^2\,\mathrm{d}\eta\,\mathrm{d}s\,\mathrm{d}s'\,.$$

Note that the internal integral is uniformly bounded by $C_G\,\varepsilon^{-2\nu}\frac{h}{\varepsilon}$. The change of variable $\eta'=\varepsilon\eta-\xi$ gives

$$\begin{aligned}\int_{\mathbb{R}^d_\eta} & e^{i\frac{s-s'}{\varepsilon}(\varepsilon^2\eta^{\cdot 2}-2\xi.\varepsilon\eta)}|\eta|^{2\nu}|f_{h,\varepsilon}(\eta)|^2\,\mathrm{d}\eta \\ &= \varepsilon^{-2\nu}\tfrac{h}{\varepsilon}e^{-i\frac{s-s'}{\varepsilon}\xi^{\cdot 2}}\int_{\mathbb{R}^d_\eta} e^{i\frac{s-s'}{\varepsilon}\eta^{\cdot 2}}|\eta+\xi|^{2\nu}\hat{G}(\eta+\xi)\,\mathrm{d}\eta\end{aligned}$$

as $f_{h,\varepsilon}(\eta)=\varepsilon^{d/2}\sqrt{\frac{h}{\varepsilon}}\hat{V}(-\varepsilon\eta)$ and $\frac{h}{\varepsilon}\,\hat{G}(\varepsilon\eta)\,\varepsilon^d=|f_{h,\varepsilon}(\eta)|^2$. For $s\neq s'$

$$\begin{aligned}\Big|\int_{\mathbb{R}^d_\eta} e^{i\frac{s-s'}{\varepsilon}\eta^{\cdot 2}}|\eta+\xi|^{2\nu}\hat{G}(\eta+\xi)\,\mathrm{d}\eta\Big| &= \big(\tfrac{\pi\varepsilon}{s'-s}\big)^{d/2}\left\|\mathcal{F}(\eta\mapsto|\eta+\xi|^{2\nu}\hat{G}(\eta+\xi))\right\|_{L^1} \\ &= \big(\tfrac{\pi\varepsilon}{s'-s}\big)^{d/2}\left\|\mathcal{F}(\eta\mapsto|\eta|^{2\nu}\hat{G}(\eta))\right\|_{L^1}\end{aligned}$$

The bound

$$\begin{aligned}\left\||\eta|^{\nu}\tilde{z}_{h,\varepsilon,t}\right\|^2_{L^2_\eta} &\le C_G\tfrac{h}{\varepsilon}\varepsilon^{-2\nu}\int_0^t\int_0^t \min\big\{\big(\tfrac{\pi\varepsilon}{s'-s}\big)^{d/2},1\big\}\,\mathrm{d}s\,\mathrm{d}s' \\ &\le C_G\tfrac{h}{\varepsilon}\varepsilon^{-2\nu}\Big[\pi^{d/2}\varepsilon^{d/2}\int_{|s-s'|\ge 2\delta,s,s'\in[0,t]}\tfrac{\mathrm{d}s\,\mathrm{d}s'}{(s'-s)^{d/2}}+2\sqrt{2}t\delta\Big] \\ &\le C_G\tfrac{h}{\varepsilon}\varepsilon^{-2\nu}\Big[\pi^{d/2}\varepsilon^{d/2}2^{d/4}2\sqrt{2}t\tfrac{2}{d-2}\delta^{1-d/2}+2\sqrt{2}t\delta\Big]\end{aligned}$$

is optimal when $\delta=\varepsilon$.

For Point (3), let $\Delta\tilde{Q}_t\,(z)=\langle z,\eta z\rangle^{\cdot 2}$. First we remark that

$$\Delta\tilde{\hat{\Psi}}_{h,\varepsilon,t} = -\frac{i}{\varepsilon}\int_0^t e^{-i\frac{t-s}{\varepsilon}\tilde{Q}^{Wick}_{h,\varepsilon}}\Delta\tilde{Q}^{Wick}\tilde{\hat{\Psi}}^{app}_{h,\varepsilon,s}\,\mathrm{d}s\,.$$

Since $i\varepsilon\partial_t\Delta\tilde{\hat{\Psi}}_t=\tilde{Q}^{Wick}\Delta\tilde{\hat{\Psi}}_t+\Delta\tilde{Q}^{Wick}\tilde{\hat{\Psi}}^{app}_t$ and that the integral expression on the right satisfies the same differential equation. The difference $\Delta\tilde{\hat{\Psi}}_{h,\varepsilon,t}$ can then be controlled as

$$\left\|\Delta\tilde{\hat{\Psi}}_{h,\varepsilon,t}\right\| \le \frac{1}{\varepsilon}\int_0^t\left\|\Delta\tilde{Q}^{Wick}\,E(\tilde{z}_{h,\varepsilon,s})\right\|\mathrm{d}s\,.$$

The relation $\langle E(z),R^{Wick}E(z)\rangle=R(z)$ with $R^{Wick}=(\Delta\tilde{Q}^{Wick})^*\Delta\tilde{Q}^{Wick}$ gives

$$\begin{aligned}&\mathrm{Symb}^{Wick}\big(\big[(\langle z,\eta z\rangle^{\cdot 2})^{Wick}\big]^2\big) \\ &\quad= (\langle z,\eta z\rangle^{\cdot 2})^2+4\varepsilon(\langle z,\eta z\rangle.\langle\eta z|)(|\eta z\rangle.\langle z,\eta z\rangle)+2\varepsilon^2(\langle\eta z|^{\cdot\otimes 2})(|\eta z\rangle^{\cdot\otimes 2})\,,\end{aligned}$$

using the estimate in Point (2), we obtain that

$$\left\|\Delta\tilde{Q}^{Wick}\,E(\tilde{z}_{h,\varepsilon,t})\right\|^2 \le C_{T_0,G,d}\big((\tfrac{ht}{\varepsilon})^4+4\varepsilon\left(\tfrac{ht}{\varepsilon}\right)^2\tfrac{ht}{\varepsilon^2}+2\varepsilon^2(\tfrac{ht}{\varepsilon^2})^2\big)$$

which gives the result for $\frac{ht}{\varepsilon}\le T_0$.

For Point (4), let $\gamma_t=\|(\varepsilon+N_\varepsilon)^{1/2}\hat{\Psi}^\sharp_t\|$, then

$$i\varepsilon\partial_t(\gamma_t^2)=\big\langle\hat{\Psi}^\sharp_t,[\Phi_\varepsilon(f_{h,\varepsilon}),N_\varepsilon]\hat{\Psi}^\sharp_t\big\rangle$$

with $f_{h,\varepsilon} = \sqrt{\frac{h}{\varepsilon}}\varepsilon^{d/2}\overline{\hat{V}(\varepsilon\eta)}$, since ξ and $\mathrm{d}\Gamma_\varepsilon(\eta)$ commute with $N_\varepsilon = \mathrm{d}\Gamma_\varepsilon(Id)$. We get

$$[a_\varepsilon(f_{h,\varepsilon}), \mathrm{d}\Gamma_\varepsilon(1)] = i\partial_s \left[\Gamma(e^{i\varepsilon s})\, a_\varepsilon(f_{h,\varepsilon})\, \Gamma(e^{-i\varepsilon s})\right]\big|_{s=0} = a_\varepsilon(\varepsilon f_{h,\varepsilon})\,.$$

The other term of the commutator can be computed analogously (but $a_\varepsilon(\cdot)$ is $\mathbb{C}$-antilinear whereas $a^*_\varepsilon(\cdot)$ is $\mathbb{C}$-linear). Introducing this relation into the differential equation and taking the modulus, we get

$$\left|i\varepsilon\partial_t(\gamma_t^2)\right| \leq \sqrt{2}^{-1}\left\|\hat{\Psi}^\sharp_t\right\|\left(\left\|a_\varepsilon(\varepsilon f_{h,\varepsilon})\,\hat{\Psi}^\sharp_t\right\| + \left\|a^*_\varepsilon(\varepsilon f_{h,\varepsilon})\,\hat{\Psi}^\sharp_t\right\|\right).$$

But

$$\left\|a^*_\varepsilon(\varepsilon f_{h,\varepsilon})\,\hat{\Psi}^\sharp_t\right\|^2 \leq \|\varepsilon f_{h,\varepsilon}\|^2_{L^2_\xi}\left\langle\hat{\Psi}^\sharp_t, (\varepsilon + N_\varepsilon)\hat{\Psi}^\sharp_t\right\rangle$$

and the same estimate holds for annihilation operators. Using $\|\hat{G}\|_{L^1} = \frac{h}{\varepsilon}\|f_{h,\varepsilon}\|^2_{L^2_\xi}$, we finally get a differential inequality for the function γ_t

$$2\varepsilon\gamma_t\partial_t\gamma_t \leq \left|i\varepsilon\partial_t\left(\gamma_t^2\right)\right| \leq \sqrt{2\varepsilon h\|\hat{G}\|_{L^1}}\,\gamma_t\,.$$

The result follows by dividing by $2\varepsilon\gamma_t$ and integrating in time, since $\gamma_0 = C_d\sqrt{\varepsilon}$. □

6. Measure of an observable at a mesoscopic scale for the approximated dynamics

6.1. **Result.** In this section we make the connection between the microscopic dynamic and the linear Boltzmann equation.

Proposition 6.1. *Let $\alpha \in [0,1)$ and assume $h^\alpha \leq \frac{ht}{\varepsilon} \leq 1$. Let $b \in \mathcal{C}^\infty_0(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$ and $\rho \in \mathcal{L}^+_1 L^2_x$, $\operatorname{Tr}\rho \leq 1$ such that the kernel of $\hat{\rho} = \operatorname{Ad}\{\mathcal{F}_x\}[\rho]$ has a bounded support. Introduce the symbol $b_t = e^{tQ}e^{2t\xi.\partial_x}b$ where Q is the collision operator introduced in Equation 3.1 with here $\sigma(\xi,\xi') = 2\pi\hat{G}(\xi'-\xi) = 2\pi|\hat{V}(\xi-\xi')|^2$. The inequality*

$$m_h(b, \rho^{\varepsilon,app}_t) \geq m_h\left(b_{\frac{ht}{\varepsilon}}, \rho\right) - \mathcal{E}_6$$

then holds with $\mathcal{E}_6 = C_{b,\mu}\frac{ht}{\varepsilon}\left(\frac{ht}{\varepsilon} + h + \left[h(\frac{ht}{\varepsilon})^{-1}\right]^{d/2-1} + h^{\mu(d,\alpha)}\right)$ for some constant $C_{b,\mu} > 0$ and $\mu\,(d,\alpha) > 0$.

Remark 6.2. This result also holds with b a symbol in $\mathcal{C}^\infty_0(\mathbb{R}^{d*}_\xi;\mathbb{C})$. The proof is the same as for Proposition 6.1, with the symplectic Fourier transform $\mathcal{F}^\sigma$ replaced by the usual Fourier transform. The special case when $b\,(\xi) = b_1(|\xi|^2)$ is of particular interest and the symbol b_t in the previous statement does not depend on t.

Proposition 6.1 is a by-product of the following stronger result.

Proposition 6.3. *Let $b_s \in \mathcal{C}^1(\mathbb{R};\mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi}))$ such that for some $R > 1$, and for all s, $\operatorname{Supp}_\xi b_s \subset B_R \setminus B_{R^{-1}}$. Let $\rho \in \mathcal{L}^+_1 L^2_x$, $\operatorname{Tr}\rho \leq 1$ such that the kernel of $\hat{\rho} = \operatorname{Ad}\{\mathcal{F}_x\}[\rho]$ has a bounded support. Then*

$$m_h\left(b_{\frac{ht}{\varepsilon}}, \rho^{\varepsilon,app}_t\right)$$
$$\geq m_h(b,\rho) - \frac{i}{\varepsilon}\int_0^t m_h\left(i\varepsilon\partial_s b_s - ih\{b_s, \xi^{\cdot 2}\} + ihQ_{-\frac{ht}{\varepsilon}}b_s, \rho^{\varepsilon,app}_s\right)\mathrm{d}s - \mathcal{E}_6\,.$$

Remark 6.4. The conservation of the support in ξ is important and is provided by the properties of the dual linear Boltzmann equation in the application of this proposition.

Proof that Proposition 6.3 implies Proposition 6.1. Since one can make mistakes between the notations of those two propositions we use notations with tildes, $\tilde{b}$ for Proposition 6.1 and without tildes for Proposition 6.3. Thus we want

$$\tilde{b} = b_{\frac{ht}{\varepsilon}}\,, \qquad \tilde{b}_{\frac{ht}{\varepsilon}} = b\,.$$

Denote by $\tilde{G}(t,t_0)$ the dynamical system associated with $(-2\xi.\partial_x - Q_{-t})_t$ given by

$$\begin{cases} \partial_t b_t = (-2\xi.\partial_x - Q_{-t})\, b_t \\ b_{t=t_0} = b_0 \end{cases}, \qquad b_t = \tilde{G}(t,t_0)\, b_0\,.$$

To have a vanishing term for b in the integral we require $b_{ht/\varepsilon} = \tilde{G}(\frac{ht}{\varepsilon},0)b$, so that with $\tilde{b}_{ht/\varepsilon} = \tilde{G}(0,-\frac{ht}{\varepsilon})\tilde{b}$, we will get the expected result. The only thing remaining to prove is $\tilde{G}(0,-t) = e^{tQ}e^{2t\xi.\partial_x}$. It is equivalent to show that $e^{2t\xi.\partial_x}\tilde{G}(t,0) = e^{-tQ}$, which is clear by derivation and using that $Q_t = e^{t2\xi.\partial_x}Qe^{-t2\xi.\partial_x}$. □

6.2. **Expression of the measure of an observable for the approximated equation.** We carry out an explicit computation using only the approximated equation.

Notation 6.5. Let $\sigma(X_1,X_2) = \xi_1.x_2 - x_1.\xi_2$ ($X_j = (x_j,\xi_j) \in \mathbb{R}^{2d}_{x,\xi}$) be the standard symplectic form on $\mathbb{R}^{2d}_{x,\xi}$.

Let $X' = (x',\xi') \in \mathbb{R}^{2d}_{x,\xi}$, the Weyl operators on L^2_x are defined by

$$\tau^h_{X'} = \left(e^{-i\sigma(\cdot,X')}\right)^W(hx,D_x) = e^{-i\sigma(\cdot,X')^W(hx,D_x)} = e^{i(\xi'\cdot hx - x'\cdot D_x)}\,,$$

their Fourier transform is denoted by $\hat{\tau}^h_P := \mathrm{Ad}\left\{\mathcal{F}_x\right\}\left[\tau^h_P\right]$. Note that the formula

$$\hat{\tau}^h_{X_1}\hat{\tau}^h_{X_2} = e^{\frac{i}{2}h\sigma(X_1,X_2)}\hat{\tau}^h_{X_1+X_2} = e^{ih\sigma(X_1,X_2)}\hat{\tau}^h_{X_2}\hat{\tau}^h_{X_1}$$

holds.

The symplectic Fourier transform $\mathcal{F}^\sigma$ on $L^2(\mathbb{R}^{2d}_{x,\xi};\mathbb{C})$ is, with $\text{đ}X = \mathrm{d}X/(2\pi)^d$,

$$\mathcal{F}^\sigma b(X) = \int_{\mathbb{R}^{2d}} e^{-i\sigma(X,X')}b(X')\text{đ}X'\,.$$

Note that $(\mathcal{F}^\sigma)^{-1} = \mathcal{F}^\sigma$.

Proposition 6.6. *Let b be a symbol in $\mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$ and $\rho \in \mathcal{L}^+_1 L^2_x$, $\mathrm{Tr}\,\rho \leq 1$, then*

$$m_h(b,\rho^{\varepsilon,app}_t) = \iiint \mathcal{F}^\sigma b(P)\, K_P(\xi_1,\xi_2)\, \hat{\tau}^h_P(\xi_2,\xi_1)\; \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \text{đ}P$$

where $K_P(\xi_1,\xi_2) = e^{-i\frac{\omega^{\xi_1}_{h,\varepsilon,t} - \omega^{\xi_2}_{h,\varepsilon,t}}{\varepsilon}}\hat{\rho}(\xi_1,\xi_2)e^{-\frac{1}{2\varepsilon}\left(|z^{\xi_1}_{h,\varepsilon,t}|^2 + |z^{\xi_2}_{h,\varepsilon,t}|^2 + 2\langle z^{\xi_2}_{h,\varepsilon,t}, e^{ip_x\cdot\varepsilon\eta}z^{\xi_1}_{h,\varepsilon,t}\rangle\right)}$.

Let $b_t \in \mathcal{C}^1(\mathbb{R};\mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi}))$, then

$$i\varepsilon\partial_t m_h(b_t,\rho^{\varepsilon,app}_t) = m_h(i\varepsilon\partial_t b_t,\rho^{\varepsilon,app}_t) + ih\left(m_{\{,\}} - m_- + m_+\right)\,.$$

where, for $\kappa = \{,\}, -, +$ we define

$$m_\kappa = \int_{\mathbb{R}^{2d}_P} \mathcal{F}^\sigma b(P)\, \mathrm{Tr}\left[\boldsymbol{\rho}^{app}_t\, \Gamma(e^{ip_x\cdot\varepsilon\eta})\, \mathscr{A}_{\kappa,P}\right]\text{đ}P$$

with the operators $\mathscr{A}_{\kappa,P}$ defined by their kernels, for $j=1,2$, by

$$\mathscr{A}_{\{,\},P}=\mathscr{A}^1_{\{,\},P}-\mathscr{A}^2_{\{,\},P}\,,\qquad ih\,\mathscr{A}^j_{\{,\},P}(\xi_1,\xi_2)=\hat{\tau}^h_P(\xi_2,\xi_1)\,\partial_t\omega_t^{\xi_j}\,,\tag{6.1}$$

$$\mathscr{A}_{-,P}=\mathscr{A}^1_{-,P}+\mathscr{A}^2_{-,P}\,,\qquad ih\,\mathscr{A}^j_{-,P}(\xi_1,\xi_2)=\hat{\tau}^h_P(\xi_2,\xi_1)\,i\partial_t\tfrac{1}{2}|z_t^{\xi_j}|^2\,,\tag{6.2}$$

$$ih\,\mathscr{A}_{+,P}(\xi_1,\xi_2)=\hat{\tau}^h_P(\xi_2,\xi_1)\,i\partial_t\,[\varphi,p_x]_2\,,\tag{6.3}$$

with $[\varphi,p_x]_2=\langle z_t^{\xi_2},e^{ip_x\cdot\varepsilon\eta}z_t^{\xi_1}\rangle$.

The indexes $\{,\}$, $-$ and $+$ are chosen to recall the terms of the linear Boltzmann equation, $\{,\}$ corresponding to $\{\xi^2,\cdot\}$, $+$ to Q_+ and $-$ to Q_-.

Remark 6.7. Each of those terms m_κ is shown in the sequel to be of the form $m_\kappa=m(c_\kappa,\rho_t^{\varepsilon,app})+\Delta_\kappa$ where Δ_κ denotes a "small" error term.

Proof. Since $b^W(hx,D_x)=\int\mathcal{F}^\sigma b(P)\,\tau^h_P\,đP$, we have for $\rho\in\mathcal{L}_1^+$

$$m_h(b,\rho)=\int\mathcal{F}^\sigma b(P)\,\mathrm{Tr}\big[\tau^h_P\,\rho\big]\,đP\,.$$

From $e^{i\varepsilon x.\lambda}\,\tau^h_P\,e^{-i\varepsilon x.\lambda}=e^{i\varepsilon\lambda.p_x}\,\tau^h_P$ and taking λ as the spectral parameter of $\mathrm{d}\Gamma_\varepsilon(D_y)$, $\Gamma(e^{i\varepsilon x.D_y})\,\tau^h_P\,\Gamma(e^{-i\varepsilon x.D_y})=\Gamma(e^{ip_x\cdot\varepsilon D_y})\,\tau^h_P$ and after conjugating with the Fourier transforms, we obtain

$$\mathrm{Ad}\,\big\{(\mathcal{F}_x\otimes\Gamma\mathcal{F}_y)\,\Gamma(e^{i\varepsilon xD_y})\big\}\,\big[\tau^h_P\big]=\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\hat{\tau}^h_P\,.$$

Thus, by translating and Fourier transforming we get the expression

$$m_h(b,\rho_t^{\varepsilon,app})=\int\mathcal{F}^\sigma b(P)\,\mathrm{Tr}\big[\hat{\boldsymbol{\rho}}_t^{app}\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\hat{\tau}^h_P\big]đP\,.$$

It then remains to compute the kernel K_P of the operator $\mathrm{Tr}_{\Gamma L^2_\eta}[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})]$ on L^2_ξ. Using $\hat{\rho}\otimes|\Omega\rangle\,\langle\Omega|=\int^\oplus_{\xi_1}\int^\oplus_{\xi_2}\hat{\rho}(\xi_1,\xi_2)\;|\Omega\rangle\,\langle\Omega|\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$ we get

$$\begin{aligned}&\operatorname*{Tr}_{\Gamma L^2_\eta}\big[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\big]\\&\quad=\operatorname*{Tr}_{\Gamma L^2_\eta}\Big[\int^\oplus_{\mathbb{R}^d_{\xi_1}}\int^\oplus_{\mathbb{R}^d_{\xi_2}}\big|E(z_t^{\xi_1})\big\rangle\big\langle E(z_t^{\xi_2})\big|e^{-i\frac{\omega_t^{\xi_1}}{\varepsilon}}\,e^{i\frac{\omega_t^{\xi_2}}{\varepsilon}}\,\hat{\rho}(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\Gamma(e^{ip_x\cdot\varepsilon\eta})\Big]\end{aligned}$$

and we obtain the kernel

$$K_P(\xi_1,\xi_2)=e^{-i\frac{\omega_t^{\xi_1}-\omega_t^{\xi_2}}{\varepsilon}}\,\hat{\rho}(\xi_1,\xi_2)\big\langle E(z_t^{\xi_2})\big|\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\big|E(z_t^{\xi_1})\big\rangle$$

which brings the expected expression using the calculus on coherent states.

For the formula for the derivative

$$\begin{aligned}i\varepsilon\partial_t m_h(b,\rho_t^{\varepsilon,app})=\iiint\Big[&\mathcal{F}^\sigma i\varepsilon\partial_t b(P)\\&+\mathcal{F}^\sigma b(P)\big\{\partial_t\big(\omega_t^{\xi_1}-\omega_t^{\xi_2}\big)-i\tfrac{1}{2}\partial_t\big(|z_t^{\xi_1}|^2+|z_t^{\xi_2}|^2\big)+i\partial_t[\varphi,p_x]_2\big\}\Big]\\&\qquad\qquad K_P(\xi_1,\xi_2)\hat{\tau}^h_P(\xi_2,\xi_1)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,đP\end{aligned}$$

and so it suffices to observe that for $\kappa = \{,\}, -, +$,

$$\begin{aligned}
&\mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\mathscr{A}_{\kappa,P}\right]\\
&\quad= \iint \hat{\rho}(\xi_1,\xi_2)\big\langle E(z_t^{\xi_2})\big|\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\big|E(z_t^{\xi_1})\big\rangle e^{-i\frac{\omega_t^{\xi_1}-\omega_t^{\xi_2}}{\varepsilon}}\mathscr{A}_{\kappa,P}(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\\
&\quad= \iint \mathscr{A}_{\kappa,P}(\xi_1,\xi_2)\,K_P(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.
\end{aligned}$$

which is the expected result. □

6.3. **Two estimates.** We need estimates to get rid of the term $\Gamma(e^{ip_x\cdot\varepsilon\eta})$ and control errors on the operators $\mathcal{A}_P$.

Proposition 6.8. *Let $\mathscr{A}_P$ be a P-dependent family of operators in $\mathcal{L}(L^2_\xi)$. Then there exists a constant $C_{G,d}$ such that*

$$\langle P\rangle^{-k}\,\big|\,\mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(\Gamma(e^{ip_x\cdot\varepsilon\eta})-\mathrm{Id}\right)\mathscr{A}_P\right]\big| \le C_{G,d}\tfrac{ht}{\varepsilon}\sup_{P\in\mathbb{R}^{2d}}\langle P\rangle^{-k}\|\mathscr{A}_P\|_{\mathcal{L}(L^2_\xi)}$$

and

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^{2d}_P}\mathcal{F}^\sigma b(P)\,\mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(\Gamma(e^{ip_x\cdot\varepsilon\eta})-\mathrm{Id}\right)\mathscr{A}_P\right]\mathrm{đ}P\Big|\\
&\qquad\qquad\le C_{G,d}\tfrac{ht}{\varepsilon}\big\|\,\langle\cdot\rangle^k\,\mathcal{F}^\sigma b\big\|_{L^1_P}\sup_P\langle P\rangle^{-k}\big\|\mathscr{A}_P\big\|_{\mathcal{L}\left(L^2_\xi\right)}.
\end{aligned}$$

This can be proved in two steps.

Remark 6.9. It suffices to prove this property with $\rho = |\psi\rangle\,\langle\psi|$ with a $\hat{\psi}$ with bounded support as any $\rho\in\mathcal{L}_1^+L^2_x$, $\mathrm{Tr}\,\rho = 1$ the decomposition $\rho = \sum_{j\ge 0}\lambda_j|\psi_j\rangle\langle\psi_j|$ holds with positive λ_j's and $\sum_j\lambda_j = 1$, and

$$\mathrm{Supp}\,\hat{\rho}(\xi,\xi')\subset B_M^2 \Leftrightarrow \forall j,\ \mathrm{Supp}\,\hat{\psi}_j\subset B_M\,.$$

Proof. For $\hat{\Psi}$ be a normed vector in $L^2_\xi\otimes\Gamma L^2_\eta$

$$\big|\,\mathrm{Tr}\left[|\hat{\Psi}\rangle\langle\hat{\Psi}|\big(\Gamma(e^{ip_x\cdot\varepsilon\eta})-\mathrm{Id}\big)\mathscr{A}_P\right]\big| \le \big\|\big(\Gamma(e^{ip_x\cdot\varepsilon\eta})-\mathrm{Id}\big)\hat{\Psi}\big\|\,\big\|\mathscr{A}_P\big\|_{\mathcal{L}(L^2_\xi)}.$$

For $\hat{\Psi} = \hat{\Psi}^{app}_{h,\varepsilon,t}$ associated with ψ, the calculus on coherent states gives

$$\begin{aligned}
\big\|\big(\Gamma(e^{ip_x\cdot\varepsilon\eta})-\mathrm{Id}\big)\hat{\Psi}^{app}_{h,\varepsilon,t}\big\|^2 &= \sup_\xi\big\|E(e^{ip_x\cdot\varepsilon\eta}z^\xi_{h,\varepsilon,t}) - E(z^\xi_{h,\varepsilon,t})\big\|^2\\
&= \sup_\xi 2\big(1-\cos\big(\tfrac{1}{\varepsilon}\Im\langle e^{ip_x\cdot\varepsilon\eta}z^\xi_{h,\varepsilon,t}, z^\xi_{h,\varepsilon,t}\rangle\big)\big) \le C^2_{G,d}(\tfrac{ht}{\varepsilon})^2\,,
\end{aligned}$$

where the inequality follows form $|1-\cos t|\le t^2/2$ and the estimates on $\|z_t\|$. We then get the second result by an integration. □

Proposition 6.10. *Let $\mathscr{E}_P$ be a P-dependent family of operators in $\mathcal{L}(L^2_\xi)$ and $\hat{\boldsymbol{\rho}}$ be a state on $L^2_\xi\otimes\Gamma L^2_\eta$. Then for any integer k (with possibly infinite quantities)*

$$\Big|\int_{\mathbb{R}^{2d}_P}\mathcal{F}^\sigma b(P)\,|\mathrm{Tr}\,[\hat{\boldsymbol{\rho}}\,\mathscr{E}_P]|\mathrm{đ}P\Big| \le \big\|\langle\cdot\rangle^k\,\mathcal{F}^\sigma b\big\|_{L^1_P}\sup_P\big\|\,\langle P\rangle^{-k}\,\mathscr{E}_P\big\|_{\mathcal{L}\left(L^2_\xi\right)}.$$

6.4. **The transport term $m_{\{,\}}$.** The result of this section is the following.

Proposition 6.11. *Let $\rho \in \mathcal{L}_1^+ L_x^2$, $\operatorname{Tr}\rho \leq 1$ and $b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ such that $\operatorname{Supp}\hat{\rho}(\xi,\xi') \subset B_R^2$, and $\operatorname{Supp}_\xi b \subset B_R$ for some $R > 0$ then*

$$m_{\{,\}} = m(-\{b, \xi^{\cdot 2}\}, t)) + \Delta_{\{,\}}$$

with $|\Delta_{\{,\}}| \leq C_{G,R,b}\,(\frac{ht}{\varepsilon} + h + (\frac{\varepsilon}{t})^{d/2})$.

Remark 6.12. We can introduce a cutoff function $\chi_R \in \mathcal{C}_0^\infty(\mathbb{R}^d_\xi)$ such that $\chi_R(B_R) = \{1\}$, $\chi_R(\mathbb{R}^d_\xi \setminus B_{R+1}) = \{0\}$ and $\chi_R(\mathbb{R}^d_\xi) \subset [0,1]$.

Proposition 6.11 is proved by doing a succession of approximations. The error terms $\Delta_{\{,\},j}$, $j = 1,2,3$ are given by the approximation process (where we write shortly b^W for $b^W(-hD_\xi,\xi)$)

$$\begin{aligned} m_{\{,\}} &= \int \mathcal{F}^\sigma b(P) \operatorname{Tr}\Big[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\tfrac{1}{ih}\left[\hat{\tau}_P^h, \chi_R\partial_t\omega\times\right]\Big]\,đP \\ &= \operatorname{Tr}\Big[\hat{\boldsymbol{\rho}}_t^{app}\,\tfrac{1}{ih}\left[b^W, \chi_R\partial_t\omega\times\right]\Big]\,đP + \Delta_{\{,\},1} \\ &= \int \mathcal{F}^\sigma\big(-\{b\,,\xi^{\cdot 2}\}\big)(P)\operatorname{Tr}\Big[\hat{\boldsymbol{\rho}}_t^{app}\,\hat{\tau}_P^h\Big]\,đP + \textstyle\sum_{j=1}^2 \Delta_{\{,\},j} \\ &= m\big(-\{b\,,\xi^{\cdot 2}\},t\big) + \textstyle\sum_{j=1}^3 \Delta_{\{,\},j}\,. \end{aligned}$$

where we used that $\mathscr{A}_{\{,\},P} = \frac{1}{ih}[\hat{\tau}_P^h\,,\,\partial_t\omega\times]$ and where the quantities $\Delta_{\{,\},j}$ are defined by

$$\begin{aligned} \Delta_{\{,\},1} &= \int \mathcal{F}^\sigma b(P)\ \operatorname{Tr}\Big[\hat{\boldsymbol{\rho}}_t^{app}\,\big(\Gamma(e^{ip_x\cdot\varepsilon\eta}) - \mathrm{Id}\big)\,\tfrac{1}{ih}\left[\hat{\tau}_P^h, \chi_R\partial_t\omega\times\right]\Big]\,đP\,, \\ \Delta_{\{,\},2} &= \operatorname{Tr}\Big[\hat{\boldsymbol{\rho}}_t^{app}\,\tfrac{1}{ih}\big(\left[b\,,\chi_R\partial_t\omega\,\times\right] - \tfrac{h}{i}\{b\,,\chi_R\xi^{\cdot 2}\}^W\big)\Big]\,đP\,, \\ \Delta_{\{,\},3} &= \int \mathcal{F}^\sigma\big(-\{b\,,\xi^{\cdot 2}\}\big)(P)\ \operatorname{Tr}\Big[\hat{\boldsymbol{\rho}}_t^{app}\big(\mathrm{Id} - \Gamma(e^{ip_x\cdot\varepsilon\eta})\big)\hat{\tau}_P^h\Big]\,đP\,. \end{aligned}$$

Proposition 6.13. *With the hypotheses and notations of Proposition 6.11, for some integer k,*

1. $|\Delta_{\{,\},1}| \leq 2\frac{ht}{\varepsilon}\|\langle\cdot\rangle^k\,\mathcal{F}^\sigma b\|_{L^1_P}\,\mathcal{O}\big(1 + h + [h(\frac{ht}{\varepsilon})^{-1}]^{d/2-1}\big)$,
2. $|\Delta_{\{,\},2}| \leq \big(\|\mathcal{F}^\sigma b\|_{L^1_P} + \|\langle\cdot\rangle^k\,\mathcal{F}^\sigma b\|_{L^1_P}\big)\,\mathcal{O}\big(h + (\frac{\varepsilon}{t})^{\frac{d}{2}-1}\big)$,
3. $|\Delta_{\{,\},3}| \leq \frac{ht}{\varepsilon}\|\mathcal{F}^\sigma\{b,\xi^{\cdot 2}\}\|_{L^1_P}$.

Proof of Proposition 6.13. Point 1 is a result of Proposition 6.8 and Lemma 6.14.
For Point 2

$$\Delta_{\{,\},2} = \int_{\mathbb{R}^{2d}_P} \mathcal{F}^\sigma b(P)\ \operatorname{Tr}\Big[\hat{\boldsymbol{\rho}}_t^{app}\,\frac{1}{ih}\Big(\big[\hat{\tau}_P^h\,,\chi_R\partial_t\omega_{h,\varepsilon,t}\times\big] - \frac{h}{i}\big\{\hat{\tau}_P^h\,,\chi_R\xi^{\cdot 2}\big\}^W\Big)\Big]\,đP$$

so that Lemma 6.14 and Proposition 6.10 give the estimation.
Point 3 is an application of Proposition 6.8. □

Lemma 6.14. *We have, for some integer k,*

$$[\hat{\tau}_P^h, \chi_R\partial_t\omega_{h,\varepsilon,t}\times] = -ih\{e^{i\sigma(P,X)},\chi_R\xi^2\}^W(-hD_\xi,\xi) + h\mathcal{O}\big(\langle P\rangle^k h + (\tfrac{\varepsilon}{t})^{\frac{d}{2}-1}\big)\,.$$

and in particular $\big\|\big[\hat{\tau}_P^h, \chi_R\partial_t\omega\times\big]\big\|_{\mathcal{L}(L^2_\xi)} \leq \langle P\rangle^k\,\mathcal{O}(h)$.

Proof of Lemma 6.14. First observe that the time derivative of ω is given by

$$\partial_t \omega_{h,\varepsilon,t} = \xi^{\cdot 2} + \Re\big\langle z^{\xi}_{h,\varepsilon,t}, f_{h,\varepsilon}\big\rangle = \xi^{\cdot 2} - h\Im \int_0^{t/\varepsilon}\int_{\mathbb{R}^d_\eta} e^{is(\eta^{\cdot 2}-2\xi.\eta)}\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s$$

once we replace $f_{h,\varepsilon}$ by its expression in terms of $\hat{V}$, use $\hat{G} = |\hat{V}|^2$ and make a change of variable. By setting

$$R(u,\xi) := \chi_R(\xi)\Im \lim_{M\to+\infty}\int_u^M \int_{\mathbb{R}^d_\eta} e^{is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s$$

we get $\chi_R\partial_t\omega = \chi_R(\xi)\xi^{\cdot 2} - hR(0,\xi) + hR(\frac{t}{\varepsilon},\xi)$. The part in $\xi^{\cdot 2}$ gives the only relevant contribution

$$[\hat{\tau}^h_P, \chi_R\xi^{\cdot 2}\times] = -ih\{e^{i\sigma(P,X)}, \chi_R\xi^{\cdot 2}\times\}^{Weyl} + \langle P\rangle^k\mathcal{O}_{h\to 0}(h^2)\,.$$

One of the other parts can be estimated without using the commutator structure

$$\big\|[\hat{\tau}^h_P, R(\tfrac{t}{\varepsilon},\xi)\times]\big\|_{\mathcal{L}(L^2_\xi)} \le 2\|\hat{\tau}^h_P\|_{\mathcal{L}(L^2_\xi)}\,\big\|R(\tfrac{t}{\varepsilon},\xi)\big\|_{L^\infty_\xi} \le C(\tfrac{\varepsilon}{t})^{\frac{d}{2}-1}$$

since

$$\int_{\mathbb{R}^d_\eta} e^{is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\hat{G}(\eta)\,\mathrm{d}\eta = e^{-is\xi^{\cdot 2}}\int_{\mathbb{R}^d_x} G(x)\,e^{-ix.\xi}\,\big(\tfrac{2\pi}{|s|}\big)^{d/2} e^{id\,\mathrm{sign}\,s\frac{\pi}{4}}e^{\frac{x^2}{2is}}\,\mathrm{d}x$$

whose modulus is bounded by $(\frac{2\pi}{|s|})^{d/2}\,\|G\|_{L^1}$.

Since $R(0,\cdot)$ is in $\mathcal{C}^\infty_0(\mathbb{R}^d_\xi)$ we can apply the symbolic calculus

$$[\hat{\tau}^h_P, hR(0,\xi)\times] = -ih^2\{e^{i\sigma(P,X)}\,, R(0,\xi)\}^W(-hD_\xi,\xi) + \mathcal{O}(h^2\langle P\rangle^k)\big)$$

where for some integer k,

$$\big\|\{e^{i\sigma(P,X)}\,, R(0,\xi)\times\}^W(-hD_\xi,\xi)\big\|_{\mathcal{L}(L^2_\xi)} = \langle P\rangle^k\mathcal{O}_{h\to 0}(1)\,,$$

which concludes the proof of the lemma. □

6.5. **The collision terms m_- and m_+.**

Proposition 6.15. *Let $b\in\mathcal{C}^\infty_0(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)$ and $\rho\in\mathcal{L}^+_1L^2_x$, $\mathrm{Tr}\,\rho\le 1$ such that for some $R>0$, $\mathrm{Supp}_\xi\, b\subset B_R - B_{1/R}$ and $\mathrm{Supp}\,\hat{\rho}(\xi,\xi')\subset B^2_R$. Then*

$$m_\pm = m(Q_{\pm,t}(b),t) + \Delta_\pm$$

and for any $\alpha\in[0,1)$, there are constants $\mu=\mu(d,\alpha)>0$ and $C_{R,b,G,d,\alpha,\mu}>0$, such that for $h^\alpha \le \frac{th}{\varepsilon}\le 1$,

$$|\Delta_\pm| \le C_{R,b,G,\mu}\big(\tfrac{ht}{\varepsilon} + h^\mu\big)\,.$$

Notation For $\zeta>0$, $r\in\mathbb{R}$ and $P\in\mathbb{R}^{2d}_{p_x,p_\xi}$, set, with $\kappa^\zeta(r) = \frac{1}{\pi}\frac{\zeta}{r^2+\zeta^2}$,

$$\mathfrak{c}(\xi) = 2\pi\int_{\mathbb{R}^d_\eta}\hat{G}(\eta+\xi)\,\delta(\eta^{\cdot 2}-\xi^{\cdot 2})\,\mathrm{d}\eta\,,$$

$$\mathfrak{c}^\zeta(\xi) = 2\pi\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,\kappa^\zeta(\eta^{\cdot 2}-2\xi.\eta)\,\mathrm{d}\eta\,,$$

$$\mathfrak{c}^\zeta_{P,t}(x,\xi) = 2\pi\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,e^{i\sigma(P,(-2t\eta,-\eta))}\kappa^\zeta(\eta^{\cdot 2}-2\xi.\eta)\,\mathrm{d}\eta\,.$$

Associate with these functions the operators defined for $b \in \mathcal{C}_0^\infty(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$ by

$$Q^\zeta_-(b) = \mathfrak{c}^\zeta\, b\,, \qquad Q_-(b) = \mathfrak{c}\, b\,,$$
$$Q^\zeta_{+,t} b(x,\xi) = \int_{\mathbb{R}^{2d}_P} \mathcal{F}^\sigma b(P)\, e^{i\sigma(P,X)}\, \mathfrak{c}^\zeta_{P,t}(x,\xi) đP\,.$$

Proposition 6.16. *For $d \geq 3$, and $h^\alpha \leq \frac{ht}{\varepsilon} \leq 1$,*

$$m_\pm = m(Q_{\pm,t}(b),t) + \sum_{k=1}^{4} \Delta_{\pm,k}$$

with

- $|\Delta_{\pm,1}| \leq \frac{ht}{\varepsilon} C_d \max\big\{\|\hat{G}\|_{L^1}, \|G\|_{L^1}\big\}\, \|\mathcal{F}^\sigma b\|_{L^1_P}$,
- $|\Delta_{\pm,2}| \leq C_{\alpha,\beta,\nu,G,d} h^\nu$,
- $|\Delta_{\pm,3}| \leq \zeta^\gamma \mathcal{N}_{k(d)}(b)\, C_{d,G,C,\gamma}$ *for* $\gamma \in (0,1)$,
- $|\Delta_{\pm,4}| \leq \frac{ht}{\varepsilon} \big\|\mathcal{F}^\sigma\big(Q_{\pm,\frac{ht}{\varepsilon}}(b)\big)\big\|_{L^1_P}$

for some ν, $\beta > 0$ with $\zeta = h^\beta$.

This result will be proved in the next paragraphs by considering successively all the error terms. These error terms $\Delta_{\pm,j}$, $j = 1, \ldots, 4$ are given by the following approximation process (where we write shortly B^W for $B^W(-hD_\xi, \xi)$)

$$\begin{aligned}
m_\pm &= \int \mathcal{F}^\sigma b(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t\, \mathscr{A}_{\pm,P}\right] đP + \Delta_{\pm,1} \\
&= \int \mathcal{F}^\sigma b(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t \left(\mathfrak{c}^\zeta_{\pm,P} e^{i\sigma(P,\cdot)}\right)^W\right] đP + \textstyle\sum_{j=1}^2 \Delta_{\pm,j} \\
&= \int \mathcal{F}^\sigma \big(Q_{\pm,\frac{ht}{\varepsilon}} b\big)(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t\, \hat{\tau}^h_P\right] đP + \textstyle\sum_{j=1}^3 \Delta_{\pm,j} \\
&= m(Q_{\pm,\frac{ht}{\varepsilon}} b, t) + \textstyle\sum_{j=1}^4 \Delta_{\pm,j}\,.
\end{aligned}$$

The error terms $\Delta_{\pm,j}$ are thus given by

$$\Delta_{\pm,1} = \int \mathcal{F}^\sigma b\,(P)\, \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t \left(\Gamma(e^{ip_x\cdot\varepsilon\eta}) - \mathrm{Id}\right) \mathscr{A}_{\pm,P}\right] đP\,, \tag{6.4}$$

$$\Delta_{\pm,2} = \int \mathcal{F}^\sigma b\,(P)\, \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t \left(\mathscr{A}_{\pm,P} - \left(\mathfrak{c}^\zeta_{\pm,P} e^{i\sigma(P,\cdot)}\right)^W\right)\right] đP\,, \tag{6.5}$$

$$\Delta_{\pm,3} = \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t \big(Q^\zeta_{\pm,\frac{ht}{\varepsilon}} b - Q_{\pm,\frac{ht}{\varepsilon}} b\big)^W\right], \tag{6.6}$$

$$\Delta_{\pm,4} = \int \mathcal{F}^\sigma \big(Q_{\pm,\frac{ht}{\varepsilon}} b\big)(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t \big(\mathrm{Id} - \Gamma(e^{ip_x\cdot\varepsilon\eta})\big) \hat{\tau}^h_P\right] đP\,, \tag{6.7}$$

since $\hat{\tau}^h_P = \big(e^{i\sigma(P,\cdot)}\big)^W$,

$$\big(Q^\zeta_{\pm,\frac{ht}{\varepsilon}} b\big)^W = \int_{\mathbb{R}^{2d}_P} \mathcal{F}^\sigma b(P) \left(\mathfrak{c}^\zeta_{\pm,P} e^{i\sigma(P,\cdot)}\right)^W đP\,,$$

and the same relation holds without ζ and

$$\int_P \mathcal{F}^\sigma \big(Q_{\pm,\frac{ht}{\varepsilon}} b\big)(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t\, \Gamma(e^{ip_x\cdot\varepsilon\eta})\, \hat{\tau}^h_P\right] đP = m\big(Q_{\pm,\frac{ht}{\varepsilon}} b, t\big)\,.$$

The term $\Delta_{\pm,4}$ can be estimated right away using Proposition 6.8.

6.5.1. *Computation of the operators $\mathscr{A}_{\pm,P}$.* We recall that the operators $\mathscr{A}_{\pm,P}$ are defined by their kernels in Equations (6.1), (6.2), (6.3).

Proposition 6.17. *The operators $\mathscr{A}_{-,j}$ can be expressed as*

$$\mathscr{A}^1_{-,P} = \int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon} \hat{\tau}^h_P \circ \Re\big(e^{is(\eta^{\cdot 2}-2\xi.\eta)}\big)\times\ \hat{G}(\eta)\,\mathrm{d}s\,\mathrm{d}\eta\,,$$

$$\mathscr{A}^2_{-,P} = \int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon} \Re\big(e^{is(\eta^{\cdot 2}-2\xi.\eta)}\big)\times\ \circ\,\hat{\tau}^h_P\,\hat{G}(\eta)\,\mathrm{d}s\,\mathrm{d}\eta\,.$$

The operator $\mathscr{A}_{+,P}$ can be decomposed as $\mathscr{A}_{+,P} = \mathscr{A}^1_{+,P} + \mathscr{A}^2_{+,P}$ with

$$\mathscr{A}^1_{+,P} = \int_0^{t/\varepsilon}\int_{\mathbb{R}^d_\eta} e^{-i\sigma\left(P,(2\frac{t}{\varepsilon}\eta,\eta)\right)}\hat{\tau}^h_P \circ e^{-is(\eta^{\cdot 2}-2\xi.\eta)}\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,,$$

$$\mathscr{A}^2_{+,P} = \int_0^{t/\varepsilon}\int_{\mathbb{R}^d_\eta} e^{-i\sigma\left(P,(2\frac{t}{\varepsilon}\eta,\eta)\right)}e^{is(\eta^{\cdot 2}-2\xi.\eta)}\circ\hat{\tau}^h_P\,\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,.$$

Proof. Computing the time derivative of $\frac{1}{2}|z_{h,\varepsilon,t}|^2$ brings

$$\partial_t \tfrac{1}{2}|z_{h,\varepsilon,t}|^2 = h\Re\int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon} e^{is(\eta^{\cdot 2}-2\xi_j.\eta)}\,\hat{G}(\eta)\,\mathrm{d}s\,\mathrm{d}\eta\,.$$

From the definition of $\mathscr{A}_{-,j,P}$ in terms of their kernel, we get

$$ih\,\mathscr{A}^1_{-,P} = i\hat{\tau}^h_P\circ(\partial_t\tfrac{1}{2}|z^\xi_t|^2)\,, \qquad ih\,\mathscr{A}^2_{-,P} = i(\partial_t\tfrac{1}{2}|z^\xi_t|^2)\circ\hat{\tau}^h_P\,,$$

hence the result for $\mathscr{A}^j_-$.

The time derivative of $[\varphi, p_x]_2$ is

$$\partial_t[\varphi,p_x]_2 = \quad h\int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon} e^{ip_x\cdot\eta}e^{is\left(\eta^{\cdot 2}-2\xi_1.\eta\right)}e^{-i\frac{t}{\varepsilon}\left(\eta^{\cdot 2}-2\xi_2.\eta\right)}\,\mathrm{d}s\,\hat{G}(\eta)\,\mathrm{d}\eta$$

$$+\,h\int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon} e^{ip_x\cdot\eta}e^{i\frac{t}{\varepsilon}\left(\eta^{\cdot 2}-2\xi_1.\eta\right)}e^{-is\left(\eta^{\cdot 2}-2\xi_2.\eta\right)}\,\mathrm{d}s\,\hat{G}(\eta)\,\mathrm{d}\eta\,.$$

We now focus on the first term (analogous computations give the second term). The definition of $\mathscr{A}_{+,P}$ in terms of their kernel gives then

$$\mathscr{A}^1_{+,P} = \int_0^{t/\varepsilon}\int_{\mathbb{R}^d_\eta} e^{ip_x\cdot\eta}e^{-i\frac{t}{\varepsilon}\left(\eta^{\cdot 2}-2\xi.\eta\right)}\circ\hat{\tau}^h_P\circ e^{is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,,$$

The relation $e^{2i\frac{t}{\varepsilon}\xi.\eta}\circ\hat{\tau}^h_P = e^{-2i\frac{t}{\varepsilon}p_\xi\eta}\hat{\tau}^h_P\circ e^{2i\frac{t}{\varepsilon}\xi.\eta}$ brings the result up to a change of variable. □

Thus we get six different terms (four for the $\mathscr{A}_-$ terms due to the real parts and two for the $\mathscr{A}_+$ terms) with a very similar structure. In order to avoid repeating analogous calculations several times we introduce the following notations.

Notation 6.18. Set (by writing shortly B^W for $B^W(-hD_\xi,\xi)$)

$$\mathscr{A}^1_{\vec\mu}(s) = \int_{\mathbb{R}^d} \hat G(\eta)\, e^{\mu_1 i\tilde\sigma}\, \hat\tau^h_P \circ e^{-\mu_2 is(\eta^{\cdot 2}-2\xi.\eta)}\,\mathrm{d}\eta\,, \tag{6.8}$$

$$\mathscr{B}^1_{\vec\mu}(s) = \int_{\mathbb{R}^d} \hat G(\eta)\, e^{\mu_1 i\tilde\sigma}\, \hat\tau^h_{(p_x-\mu_2 2s\eta,p_\xi)}\, e^{-\mu_2 is\eta^{\cdot 2}}\,\mathrm{d}\eta\,, \tag{6.9}$$

$$\mathscr{C}^{1,\zeta}_{\vec\mu} = \int_{\mathbb{R}^d} \hat G(\eta)\big(e^{\mu_1 i\tilde\sigma} e^{i\sigma(P,\cdot)}\big)^W \frac{\mathrm{d}\eta}{\zeta+\mu_2 i\,(\eta^{\cdot 2}-2\xi.\eta)}\,, \tag{6.10}$$

$$\mathscr{A}^2_{\vec\mu}(s) = \int_{\mathbb{R}^d} \hat G(\eta)\, e^{\mu_1 i\tilde\sigma} e^{\mu_2 is(\eta^{\cdot 2}-2\xi.\eta)} \circ \hat\tau^h_P\,\mathrm{d}\eta\,, \tag{6.11}$$

$$\mathscr{B}^2_{\vec\mu}(s) = \int_{\mathbb{R}^d} \hat G(\eta)\, e^{\mu_1 i\tilde\sigma}\, \hat\tau^h_{(p_x+\mu_2 2s\eta,p_\xi)}\, e^{\mu_2 is\eta^{\cdot 2}}\,\mathrm{d}\eta\,, \tag{6.12}$$

$$\mathscr{C}^{2,\zeta}_{\vec\mu} = \int_{\mathbb{R}^d} \hat G(\eta)\big(e^{\mu_1 i\tilde\sigma} e^{i\sigma(P,\cdot)}\big)^W \frac{\mathrm{d}\eta}{\zeta-\mu_2 i\,(\eta^{\cdot 2}-2\xi.\eta)}\,, \tag{6.13}$$

with $\tilde\sigma = \sigma\big(P,(-2h\frac{t}{\varepsilon}\eta,-\eta)\big)$. The terms μ_1, μ_2 are chosen to adapt to the cases of the terms $m_\pm$. More precisely, for $j=1$, 2, the previous quantities become

$$\mathscr{A}^j_- = \int_0^{t/\varepsilon} \big(\mathscr{A}^j_{0,1}(s)+\mathscr{A}^j_{0,-1}(s)\big)\,\mathrm{d}s \qquad \text{and} \qquad \mathscr{A}^j_+ = \int_0^{t/\varepsilon} \mathscr{A}^j_{1,1}(s)\,\mathrm{d}s\,.$$

We first show that the operators $\mathscr{C}^\zeta_{\vec\mu}$ are good approximations of the operators $\mathscr{A}_{\vec\mu} = \int_0^{t/\varepsilon}\mathscr{A}_{\vec\mu}(s)\,\mathrm{d}s$ if the parameter ζ is well chosen. We use the operators $\int_0^{t/\varepsilon}\mathscr{B}_{\vec\mu}(s)\,\mathrm{d}s$ as an intermediate step. Then we study the limit of the operators $\mathscr{C}^\zeta_{\vec\mu}$, with a distinction between the cases m_- and m_+.

6.5.2. *Estimate of the error terms $\Delta_{\pm,1}$.*

Proposition 6.19. *For $d\geq 3$,*

$$|\Delta_{\pm,1}| \leq \tfrac{ht}{\varepsilon} C_d \max\big\{\|\hat G\|_{L^1},\|G\|_{L^1}\big\}\,\|\mathcal{F}^\sigma b\|_{L^1_P}\,.$$

Proof. The term $\Delta_{\pm,1}$ was defined in Equation (6.4). This inequality follows from Propositions 6.8 and 6.20 below since $s\mapsto\min\{1,s^{-d/2}\}$ is integrable on $\mathbb{R}^+$ for $d\geq 3$. □

Proposition 6.20. *The families of operators $\mathscr{A}(s)=\mathscr{A}^j_{\vec\mu}(s)$ satisfy*

$$\|\mathscr{A}(s)\|_{\mathcal{L}(L^2_\xi)} \leq C_d \max\big\{\|\hat G\|_{L^1},\|G\|_{L^1}\big\}\min\big\{1,s^{-d/2}\big\}\,.$$

Proof. A uniform estimate of Equations (6.8) and (6.11) yields $\|\mathscr{A}^j_{\vec\mu}(s)\|_{\mathcal{L}(L^2_\xi)}\leq C_d\|\hat G\|_{L^1}$. In order to obtain the part of the estimate with the dependence in s, we use the formula

$$\big\|\mathscr{A}^j_{\vec\mu}(s)\big\|_{\mathcal{L}(L^2_\xi)} = \sup\big\{\big|\big\langle\psi,\mathscr{A}^j_{\vec\mu}(s)\varphi\big\rangle\big|\,,\ \|\psi\|_{L^2_\xi}=\|\varphi\|_{L^2_\xi}=1\big\}\,.$$

We can then compute, for $\psi,\, \varphi \in L^2_\xi$,

$$\begin{aligned}\left\langle \psi, \mathscr{A}^j_{\vec{\mu}}(s)\varphi\right\rangle &= \int_{\mathbb{R}^d_\eta} \left\langle \psi, \hat{G}(\eta)\, e^{i\mu_1\tilde{\sigma}}\, \hat{\tau}^h_P \circ e^{-\mu_2 is(\eta^{\cdot 2}-2\xi.\eta)}\varphi\right\rangle_\xi \mathrm{d}\eta \\ &= \int_{\mathbb{R}^d_\xi} \left\langle \hat{G}(\eta)\, \hat{\tau}^h_{-P}\psi(\xi), e^{\mu_1 i\tilde{\sigma}} e^{-\mu_2 is(\eta^{\cdot 2}-2\xi.\eta)}\varphi(\xi)\right\rangle_\eta \mathrm{d}\xi \\ &= \int_{\mathbb{R}^d_\theta} \langle \psi_\theta, \varphi_{\vec{\mu},\theta}\rangle_\xi \,\mathrm{d}\theta/(2\pi)^d\,,\end{aligned}$$

where we defined, for $\theta \in \mathbb{R}^d_\theta$,

$$\varphi_{\vec{\mu},\theta} = \int e^{i\theta\eta} e^{\mu_1 i\tilde{\sigma}} e^{-\mu_2 is(\eta^{\cdot 2}-2\xi.\eta)}\varphi(\xi)\,\mathrm{d}\eta\,,\quad \psi_\theta = \int e^{i\theta\eta}\,\hat{G}(\eta)\,\hat{\tau}^h_{-P}\,\psi(\xi)\,\mathrm{d}\eta\,.$$

We first compute

$$\varphi_{\vec{\mu},\theta}(\xi) = (\tfrac{\pi}{s})^{d/2} e^{i\frac{(\theta+\mu_2 2s\xi+\mu_1(2hsp_\xi - p_x))^2}{4\mu_2 s}} e^{i\frac{\pi}{4}d}\varphi(\xi)$$

where we used the formula $\int e^{-ix\eta}e^{-a\eta^2}\,\mathrm{d}\eta = \left(\frac{\pi}{a}\right)^{d/2} e^{-x^2/4a}$ with $a = \mu_2 is$ and $x = -\left(\theta + \mu_2 2s\xi + \mu_1\left(2hsp_\xi - p_x\right)\right)$ and so $\|\varphi_{\vec{\mu},\theta}\|_{L^\infty(\mathbb{R}^d_\theta;L^2_\xi)} \le \left(\frac{\pi}{s}\right)^{d/2}\|\varphi\|_{L^2_\xi}$. We now observe that

$$\Big\| \int e^{i\theta\eta}\,\hat{G}(\eta)\,\hat{\tau}^h_{-P}\,\mathrm{d}\eta\Big\|_{L^1(\mathbb{R}^d_\theta;\mathcal{L}(L^2_\xi))} \le (2\pi)^d\,\|G\|_{L^1}$$

so that $\|\psi_\theta\|_{L^1(\mathbb{R}^d_\theta;L^2_\xi)} \le C_d\,\|G\|_{L^1}\,\|\psi\|_{L^2_\xi}$. And finally

$$\left|\left\langle \psi, \mathscr{A}_{\vec{\mu}}(s)\varphi\right\rangle\right| \le C_d\ \|G\|_{L^1}\ (\tfrac{\pi}{s})^{d/2}\,\|\varphi\|_{L^2_\xi}\,\|\psi\|_{L^2_\xi}$$

and we obtain the desired result $\|\mathscr{A}_{\vec{\mu}}(s)\|_{\mathcal{L}(L^2_\xi)} \le C_d\,\|G\|_{L^1}\,s^{-d/2}\,.$ □

6.5.3. *Estimate of the error terms $\Delta_{\pm,2}$.*

Proposition 6.21. *Let $\alpha \in (0,1]$. There are constants $\beta = \beta(d,\alpha) \in (0,1)$, $\nu = \nu(d,\alpha) \in (0,1)$ and $C = C(\alpha,\beta,\nu,d,G) > 0$ such that, for $h^\alpha \le \frac{th}{\varepsilon} \le 1$, and $\zeta = h^\beta$,*

$$|\Delta_{\pm,2}| \le \|\left\langle\cdot\right\rangle^k \mathcal{F}^\sigma b\|_{L^1} C h^\nu\,.$$

In order to prove this result we use Proposition 6.10 and thus control

$$\Big\| \int_0^{t/\varepsilon} \mathscr{A}(s)\,\mathrm{d}s - \mathscr{C}^\zeta\Big\|_{\mathcal{L}(L^2_\xi)}\,.$$

We first give an abstract result and then show that our cases fit within this framework.

Proposition 6.22. *For M, t, ε such that $1 \le M \le \frac{t}{\varepsilon}$. Suppose given $(\mathscr{A}(s))_{s\ge 0}$, $(\mathscr{B}(s))_{s\ge 0}$ and $(\mathscr{C}^\zeta)_{0<\zeta<1}$ three families of operators in $\mathcal{L}(L^2_\xi)$ (also dependent on h and $P = (p_x,p_\xi)$) such that for some constants $C_\mathscr{A}$, $C_{\mathscr{A},\mathscr{B}}$, $C_{\mathscr{B},\mathscr{C}}$, independent of $h,\varepsilon,t,P,M,\zeta$,*

1. $\|\mathscr{A}(s)\|_{\mathcal{L}(L^2_\xi)} \le C_\mathscr{A}\min\left\{1, s^{-d/2}\right\}$,
2. $\|\mathscr{A}(s) - \mathscr{B}(s)\|_{\mathcal{L}(L^2_\xi)} \le C_{\mathscr{A},\mathscr{B}}hs\,|p_\xi|$,

(3) $r_{\zeta,M}(x,\xi) := \mathrm{Symb}^{Weyl}(\int_0^M \mathscr{B}(s)\, e^{-\zeta s}\,\mathrm{d}s - \mathscr{C}^\zeta)$ *satisfies for some* $k = k(d) \in \mathbb{N}$,

$$\sup_{|\alpha|\le k} \|\partial^\alpha_{x,\xi} r_{\zeta,M}\|_{L^\infty_{x,\xi}} \le C_{\mathscr{B},\mathscr{C}} \langle P\rangle^k \left(\tfrac{M}{\zeta}\right)^k e^{-\zeta M}\,.$$

Then, for $\zeta M \ge 1$,

(1) $\| \int_0^{t/\varepsilon} \mathscr{A}(s)\,\mathrm{d}s\|_{\mathcal{L}(L^2_\xi)} \le \frac{d}{d-2} C_{\mathscr{A}}$,

(2) $\| \int_0^{t/\varepsilon} \mathscr{A}(s)\,\mathrm{d}s - \int_0^M \mathscr{A}(s)\,\mathrm{d}s\|_{\mathcal{L}(L^2_\xi)} \le \frac{2}{d-2} C_{\mathscr{A}} M^{1-\frac{d}{2}}$,

(3) *for* $d \ge 3$,

$$\Big\| \int_0^M \mathscr{A}(s)\big(1 - e^{-\zeta s}\big)\,\mathrm{d}s \Big\|_{\mathcal{L}(L^2_\xi)} \le 5 C_{\mathscr{A}} \zeta^{1/2}$$

(4) $\| \int_0^M (\mathscr{A}(s) - \mathscr{B}(s))\, e^{-\zeta s}\,\mathrm{d}s\|_{\mathcal{L}(L^2_\xi)} \le \frac{1}{2} C_{\mathscr{A},\mathscr{B}} h \zeta^{-2} |p_\xi|$,

(5) *for some integer* $k = k(d)$,

$$\Big\| \int_0^M \mathscr{B}(s)\, e^{-\zeta s}\,\mathrm{d}s - \mathscr{C}^\zeta \Big\|_{\mathcal{L}(L^2_\xi)} \le C_{d,k'} C_{\mathscr{B},\mathscr{C}} \langle P\rangle^k \left(\tfrac{M}{\zeta}\right)^k e^{-\zeta M}\,.$$

(6) *Let* $\frac{ht}{\varepsilon} \ge h^\alpha$, $\zeta = h^\beta$ *with* $\beta \in (0, \frac{1}{2})$ *and* $\beta + \alpha < 1$, *and* $\nu = \nu(\alpha,\beta) < \min\{(1-\alpha)/2, \beta/2, 1-2\beta\}$, *we have*

$$\Big\| \int_0^{\frac{t}{\varepsilon}} \mathscr{A}(s)\,\mathrm{d}s - \mathscr{C}^\zeta \Big\|_{\mathcal{L}(L^2_\xi)} \le C h^\nu$$

with $C = C(\nu, \alpha, \beta, C_{\mathscr{A}}, C_{\mathscr{A}\mathscr{B}}, C_{\mathscr{B}\mathscr{C}})$.

Proof. Points 1 and 2 are proved by integration of the first assumed estimate and using $1 \le M \le \frac{t}{\varepsilon}$ for *2*.

Point 3 is proved by integration of the first assumed estimate, using $1 - e^{-\zeta s} \le \zeta s$ for $\zeta s \le 1$ and $1 - e^{-\zeta s} \le 1$ for $\zeta s \ge 1$,

$$\int_0^M (1 - e^{-\zeta s}) \min\{1, s^{-d/2}\}\,\mathrm{d}s \le \zeta \int_0^1 s\,\mathrm{d}s + \zeta \int_1^{1/\zeta} s^{1-\frac{d}{2}}\,\mathrm{d}s + \int_{1/\zeta}^{+\infty} s^{-d/2}\,\mathrm{d}s\,,$$

which brings the result.

For Point 4, we use the second assumption and $\int_0^M s e^{-\zeta s}\,\mathrm{d}s \le \zeta^{-2} \int_0^{+\infty} u e^{-u}\,\mathrm{d}u$.

For Point 5, the known estimates for pseudo-differential operators give

$$\big\| r^W(-hD_\xi, \xi) \big\| \le C_k \sup_{|\alpha|\le N_k} \big\| \partial^\alpha_{x,\xi} r \big\|_{L^\infty(\mathbb{R}^{2d})}\,.$$

This and the third hypothesis imply the result.

For Point 6, we would like to choose the (h-dependent) parameters M and ζ such that the quantity

$$M^{1-\frac{d}{2}} + \sqrt{\zeta} + h\zeta^{-2} + \left(\tfrac{M}{\zeta}\right)^k e^{-\zeta M}\,,$$

is small when h tends to 0 and M not too big. We choose $hM = h^\alpha$ and $\zeta = h^\beta$ with $\beta + \alpha < 1$, $\alpha, \beta > 0$ so that the previous quantity is smaller than

$$h^{(1-\alpha)\left(\frac{d}{2}-1\right)} + h^{\beta/2} + h^{1-2\beta} + h^{-k(1-\alpha+\beta)} \exp\big(-(h^{\beta+\alpha-1})\big)\,.$$

In order to get a small quantity it suffices to require $\beta < \frac{1}{2}$. Then we get an error term whose size is controlled by $h^{\nu(\alpha,\beta)}$. $\square$

Proposition 6.23. *The families of operators $\mathscr{A}(s)=\mathscr{A}^{j}_{\vec{\mu}}(s)$, $\mathscr{B}(s)=\mathscr{B}^{j}_{\vec{\mu}}(s)$ and $\mathscr{C}^{\zeta}=\mathscr{C}^{j,\zeta}_{\vec{\mu}}$ satisfy the hypotheses of Proposition 6.22 with*

$$C_{\mathscr{A}}=C_d\max\left\{\|\hat{G}\|_{L^1},\|G\|_{L^1}\right\},\ C_{\mathscr{A},\mathscr{B}}=\left\||\cdot|\,\hat{G}\right\|_{L^1},\ C_{\mathscr{B},\mathscr{C}}=\left\|\langle\cdot\rangle^k\,\hat{G}\right\|_{L^1},$$

for some integer k.

Proof. Point 1 is contained in Proposition 6.20.

We show Point 2 for $\mathscr{A}^1_{\vec{\mu}}$ and $\mathscr{B}^1_{\vec{\mu}}$, the proof can be adapted to the case of $\mathscr{A}^2_{\vec{\mu}}$ and $\mathscr{B}^2_{\vec{\mu}}$. We observe that

$$\hat{\tau}^h_P\circ\left(e^{\mu_2 is2\xi.\eta}\times\right)=e^{-\mu_2 is\eta hp_\xi}\hat{\tau}^h_{P-(\mu_2 2s\eta,0)}$$

and

$$\left(e^{i\sigma(P,X)}e^{\mu_2 is2\xi.\eta}\right)^W(-hD_\xi,\xi)=\hat{\tau}^h_{(p_x-\mu_2 2s\eta,p_\xi)}\,.$$

Thus we obtain the estimation

$$\left\|\hat{\tau}^h_P\circ\left(e^{\mu_2 is2\xi.\eta}\times\right)-\left(e^{i\sigma(P,X)}e^{\mu_2 is2\xi.\eta}\right)^W(-hD_\xi,\xi)\right\|_{\mathcal{L}(L^2_\xi)}\le hs\,|\eta|\,|p_\xi|$$

Since the Weyl symbol of $\mathscr{B}^1_{\vec{\mu}}(s)$ is

$$\frac{1}{2}\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,e^{i\mu_1\tilde{\sigma}}e^{i\sigma(P,X)}e^{-\mu_2 is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\mathrm{d}\eta$$

we get the estimate with $C_{\mathcal{A},\mathcal{B}}=\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,|\eta|\,\mathrm{d}\eta$.

For Point 3, the Weyl symbol of $\int_0^M\mathscr{B}^1_{\vec{\mu}}(s)\,e^{-\zeta s}\,\mathrm{d}s$ is

$$\begin{aligned}\mathrm{Symb}^{Weyl}&\int_0^M\mathscr{B}^1_{\vec{\mu}}(s)\,e^{-\zeta s}\,\mathrm{d}s\\&=\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,e^{\mu_1 i\tilde{\sigma}}e^{i\sigma(P,X)}\left[\frac{e^{-\mu_2 is\left(\eta^{\cdot 2}-2\xi.\eta\right)-\zeta s}}{-\mu_2 i\left(\eta^{\cdot 2}-2\xi.\eta\right)-\zeta}\right]_0^M\mathrm{d}\eta\\&=\mathrm{Symb}^{Weyl}\,\mathscr{C}^{1,\zeta}_{\vec{\mu}}+r_{\zeta,M}\end{aligned}$$

with

$$r_{\zeta,M}(x,\xi)=-\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,e^{\mu_1 i\tilde{\sigma}}e^{i\sigma(P,X)}\frac{e^{-\mu_2 iM\left(\eta^{\cdot 2}-2\xi.\eta\right)-\zeta M}}{\mu_2 i\left(\eta^{\cdot 2}-2\xi.\eta\right)+\zeta}\,\mathrm{d}\eta\,.$$

and this expression allows us to get the estimate

$$\left|\partial^\alpha_{x,\xi}r_{\zeta,M}(x,\xi)\right|\le\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,\langle P\rangle^k(M\langle\eta\rangle)^k\frac{1}{\zeta^{k+1}}e^{-\zeta M}\,\mathrm{d}\eta$$

which yields the result with $k+1$ replaced by k. The same proof holds for $\mathscr{B}^2_{\vec{\mu}}(s)$ and $\mathscr{C}^{2,\zeta}_{\vec{\mu}}$. ◻

6.5.4. *Estimate of the error term $\Delta_{-,3}$.*

Proposition 6.24. *Let $b\in\mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$ with $\mathrm{Supp}_\xi\,b\subset B_R\setminus B_{1/R}$ for some $R>1$. Let $\gamma\in(0,1)$. There exists a constant $C_{G,b,\gamma}>0$ such that, for all $\zeta>0$,*

$$|\Delta_{-,3}|\le\zeta^\gamma\mathcal{N}_k(b)C_{G,b,\gamma}$$

for some integer $k=k\,(d)$ big enough.

Proof. We recall that

$$\Delta_{-,3} = \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}(Q_-^\zeta b - Qb)^W(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta))\right]$$

so that

$$|\Delta_{-,3}| \le \left\|(Q_-^\zeta b - Q_- b)^W(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta))\right\|_{\mathcal{L}(L^2_\xi \otimes \Gamma L^2_\eta)} \le C_{k,d}\mathcal{N}_k(Q_-^\zeta b - Q_- b)$$

for some integer k big enough. By recalling $Q_-^\zeta(b) = \mathfrak{c}^\zeta\, b$ and $Q_-(b) = \mathfrak{c}\, b$ it is then sufficient to prove Lemma 6.25 below. $\square$

Lemma 6.25. *For any integer k and γ in $[0,1)$, a positive constant $C_{k,\gamma,G,C}$ exists such that for $\zeta \in (0,\zeta_0)$*

$$\sup_{|\alpha|\le k}\ \sup_{|\xi|\in[R^{-1},R]} \left|\partial_\xi^\alpha\left(\mathfrak{c}^\zeta - \mathfrak{c}\right)(\xi)\right| \le C_{k,\gamma,G,R}\zeta^\gamma\,.$$

Proof. With κ^ζ, $\mathfrak{c}$, $\mathfrak{c}^\zeta$ introduced in Definition 6.5, $\mathfrak{c}^\zeta - \mathfrak{c}$ can be expressed as

$$\left(\mathfrak{c}^\zeta - \mathfrak{c}\right)(\xi) = \int_{\mathbb{R}^d_\eta} \hat{G}(\eta)\,\kappa^\zeta(\eta^{\cdot 2} - 2\xi.\eta)\,\mathrm{d}\eta - \int_{\mathbb{R}^d_\eta} \hat{G}(\xi+\eta)\,\delta\big(|\eta|^2 - |\xi|^2\big)\,\mathrm{d}\eta\,.$$

We express the first integral as

$$\begin{aligned}\int_{\mathbb{R}^d_\eta} \hat{G}(\eta)\,\kappa^\zeta\big((\eta-\xi)^{\cdot 2} - \xi^{\cdot 2}\big)\,\mathrm{d}\eta &= \int_{S^{d-1}}\int_{\mathbb{R}_\rho} f_{\xi,\omega}(r)\,\kappa^\zeta(\xi^{\cdot 2} - r)\,\mathrm{d}r\,\mathrm{d}\omega \\ &= \int_{S^{d-1}} f_{\xi,\omega} * \kappa^\zeta(\xi^{\cdot 2})\,\mathrm{d}\omega\end{aligned}$$

and $f_{\xi,\omega}(r) := \frac{1}{2}r^{\frac{d-2}{2}}g(\xi + \sqrt{r}\omega)\,1_{[0,+\infty)}(r)$. The partial derivative

$$\partial_{\xi_j} f_{\xi,\omega}(r) = \frac{1}{2}r^{\frac{d-2}{2}}\,\partial_{\xi_j} g(\xi+\sqrt{r}\omega)\,1_{[0,+\infty)}(r)$$

has the same form as the function f_ξ. Then we observe that

$$\begin{aligned}&\partial_{\xi_j}\big(f_{\xi,\omega} * \kappa^\zeta - f_{\xi,\omega}\big)\big(|\xi|^2\big) \\ &\qquad = \left[(\partial_{\xi_j} f_{\xi,\omega}) * \kappa^\zeta - \partial_{\xi_j} f_{\xi,\omega}\right]\big(|\xi|^2\big) + \left[\partial_r(f_{\xi,\omega} * \kappa^\zeta - f_{\xi,\omega})\right]\big(|\xi|^2\big)\,2\xi_j\end{aligned}$$

so that by doing successive derivations it suffices to deal only with quantities of the form $\partial_r^k(\partial_\xi^\beta f_{\xi,\omega} * \kappa^\zeta - \partial_\xi^\beta f_{\xi,\omega})$ which are in fact of the form $\partial_r^k(f * \kappa^\zeta - f)$ with f satisfying the hypotheses of Lemma 6.26 uniformly in ω so that we get the expected control, by integration over ω. $\square$

Lemma 6.26. *Let $f : \mathbb{R}_r \to \mathbb{R}$ continuous, vanishing on $\mathbb{R}^-$, such that $f|_{\mathbb{R}^+_*} \in \mathcal{C}^\infty(\mathbb{R}^+_*)$ is rapidly decreasing towards $+\infty$. Let $0 < r_{\min} < r_{\max}$. Then*

$$\forall\gamma\in(0,1)\,,\quad \exists C_{f,\gamma}\,,\qquad \left\|\,\partial_r^k[f * \kappa^\zeta - f]\big|_{[r_{\min},r_{\max}]}\right\|_{L^\infty} \le C_\gamma\zeta^\gamma\,.$$

Proof. We choose A and Δr such that $0 < A < \Delta r < r_{\min}/2$. Let χ_1 a $\mathcal{C}^\infty$ decreasing function such that

$$\begin{aligned}\chi_1(r) &= 1 \quad \text{if} \quad r \le A/2 \\ &= 0 \quad \text{if} \quad A \le r\,.\end{aligned}$$

Let $f_1 = \chi_1 f$ and $f_2 = (1-\chi_1)\,f$ then

$$f * \delta^\zeta = f_1 \underset{\mathcal{E}',\mathcal{C}^\infty}{*} \kappa^\zeta + f_2 \underset{\mathcal{S},L^1}{*} \kappa^\zeta\,.$$

Since $\partial_r^k\left(f_2 * \kappa^\zeta\right) = \left(\partial_r^k f_2\right) * \kappa^\zeta$, Lemma 6.30 gives, for the second term,

$$\left\|\left(\partial_r^k f_2\right) * \kappa^\zeta - \pi \partial_r^k f_2\right\|_{L^\infty} \leq C_\gamma\left(\left\|f_2^{(k)}\right\|_\infty + \left\|f_2^{(k+1)}\right\|_\infty\right)\zeta^\gamma .$$

We are only interested in $r \in [r_{\min}, r_{\max}]$ with $0 < r_{\min} < r_{\max}$ when evaluating $\partial_r^k(f * \kappa^\zeta)$. We insert another cutoff function $\chi_2 \in \mathcal{C}_0^\infty(\mathbb{R})$ such that

$$\begin{aligned} \chi_2(r) &= 0 \quad \text{if} \quad && r \leq r_{\min} - 2\Delta r \\ &= 1 \quad \text{if} \quad && r_{\min} - \Delta r \leq r \leq r_{\max} + \Delta r \\ &= 0 \quad \text{if} \quad && r_{\max} + 2\Delta r \leq r \end{aligned}$$

Then $f_1 * \kappa^\zeta = f_1 * \chi_2 \kappa^\zeta + f_1 * (1-\chi_2)\,\kappa^\zeta$ and our hypotheses on the supports give

$$\begin{aligned} \operatorname{Supp}\{f_1 * (1-\chi_2)\kappa^\zeta\} &\subset \operatorname{Supp} f_1 + \operatorname{Supp}(1-\chi_2) \\ &\subset \mathbb{R} \setminus [r_{\min} - \Delta r + A, r_{\max} + \Delta r] . \end{aligned}$$

Since $A < \Delta r$ we obtain $\left[f_1 * (1-\chi_2)\,\kappa^\zeta\right]\big|_{[r_{\min}, r_{\max}]} = 0$ and we can restrict ourselves to the computation of $f_1 \underset{\mathcal{E}', \mathcal{C}_0^\infty}{*} \chi_2 \kappa^\zeta$. More precisely we want to estimate

$$\left\|\partial_r^k\left(f_1 \underset{\mathcal{E}', \mathcal{C}_0^\infty}{*} \chi_2 u_\zeta\right)\Big|_{[r_{\min}, r_{\max}]}\right\|_{L^\infty}$$

since $\chi_2 \delta = 0$ and thus $f_1 \underset{\mathcal{E}', \mathcal{E}'}{*} \chi_2 \delta = 0$. But the same considerations hold for the supports of the derivatives. Thus it is sufficient to observe that we have the control

$$\left\|f_1 \underset{L^1, \mathcal{C}_0^\infty}{*} \partial_r^k(\chi_2 \kappa^\zeta)\right\|_{L^\infty} \leq \|f_1\|_{L^1} \|\partial^k(\chi_2 \kappa^\zeta)\|_{L^\infty} \leq \|f_1\|_{L^1} C_{\chi_2} \sup_{r \geq r_{\min} - 2\Delta r} |\partial^k \kappa^\zeta|$$

where the sup is controlled by $C\zeta$ with C only dependent on Δr and $r_{\min}$ since

$$2\partial^k \kappa^\zeta(r) = i^k k! \frac{-\left(ir - \zeta\right)^{k+1} + \left(ir + \zeta\right)^{k+1}}{\left(r^2 + \zeta^2\right)^{k+1}} .$$

Consequently

$$\left\|\partial_r^k\left[f_1 \underset{\mathcal{E}', \mathcal{C}_0^\infty}{*} \chi_2 \kappa^\zeta - f_1 \underset{\mathcal{E}', \mathcal{E}'}{*} \chi_2 \delta\right]\Big|_{[r_{\min}, r_{\max}]}\right\|_{L^\infty} \leq C\zeta$$

and this ends the proof. □

6.5.5. *Estimate of the error term $\Delta_{+,3}$.*

Remark 6.27. Throughout this section we will make definitions that are dependent on the value of $\frac{th}{\varepsilon}$. This will not be a problem as long as $\frac{th}{\varepsilon} \leq 1$ which will be satisfied with our choice of $\varepsilon = \varepsilon(h) \gg h$.

Proposition 6.28. *Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ with $\operatorname{Supp}_\xi b \subset B_R \setminus B_{1/R}$ for some $R > 1$. Let $\gamma \in (0,1)$. There exists a constant $C_{G,R,\gamma} > 0$ such that, for all $\zeta > 0$,*

$$|\Delta_{+,3}| \leq \zeta^\gamma \, \mathcal{N}_k(b) \, C_{G,R,\gamma}$$

for some integer $k = k(d)$ big enough.

Proof. Since $\Delta_{+,3} = \operatorname{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(Q^\zeta_{+,\frac{ht}{\varepsilon}} b - Q_{+,\frac{ht}{\varepsilon}} b\right)^W(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta))\right]$ we get

$$\begin{aligned} |\Delta_{+,3}| &\leq \left\|\left(Q^\zeta_{+,\frac{ht}{\varepsilon}} b - Q_{+,\frac{ht}{\varepsilon}} b\right)^W(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta))\right\|_{\mathcal{L}(L^2_\xi \otimes \Gamma L^2_\eta)} \\ &\leq C_{k,d} \, \mathcal{N}_k\left(Q^\zeta_{+,\frac{ht}{\varepsilon}} b - Q_{+,\frac{ht}{\varepsilon}} b\right) \end{aligned}$$

for some integer $k = k(d)$ big enough.

Thus we boil down to prove that for any integer $k \geq 0$ there is a constant $C_{k,b,G,\gamma} > 0$ such that for any $\zeta > 0$

$$\mathcal{N}_k\big(Q^{\zeta}_{+,\frac{ht}{\varepsilon}} b - Q_{+,\frac{ht}{\varepsilon}} b\big) \leq C_{k,G,\gamma}\, \mathcal{N}_k(b)\, \zeta^{\gamma}\,.$$

But we have a convenient expression for $Q^{\zeta}_{+,\frac{ht}{\varepsilon}}$

$$\begin{aligned} Q^{\zeta}_{+,\frac{ht}{\varepsilon}} b(x,\xi) &= 2\pi \int_{\mathbb{R}^d_\eta} \hat{G}(\eta)\, b\big(x - 2\tfrac{ht}{\varepsilon}\eta, \xi - \eta\big)\, \kappa^{\zeta}(\eta^{\cdot 2} - 2\xi.\eta)\, \mathrm{d}\eta \\ &= 2\pi \int_{\mathbb{R}^d_\eta} \hat{G}(\xi - \eta)\, b\big(x - 2\tfrac{ht}{\varepsilon}\xi + 2\tfrac{ht}{\varepsilon}\eta, \eta\big)\, \kappa^{\zeta}(\eta^{\cdot 2} - 2\xi.\eta)\, \mathrm{d}\eta \\ &= \pi \int_{\mathbb{S}^{d-1}_\omega} \int_{\mathbb{R}^+_r} \varphi_\omega(x,\xi,r)\, K^{\zeta}(r - \xi^{\cdot 2})\, \mathrm{d}r\, \mathrm{d}\omega\,, \end{aligned}$$

with $\varphi_\omega(x,\xi,r) = 0$ for $r \leq 0$, and for $r \geq 0$,

$$\varphi_\omega(x,\xi,r) = \hat{G}(\xi - \sqrt{r}\omega)\, b\big(x - 2\tfrac{ht}{\varepsilon}\xi + 2\tfrac{ht}{\varepsilon}\sqrt{r}\omega, \sqrt{r}\omega\big)\, r^{d/2-1} \tag{6.14}$$

defined for $\omega \in \mathbb{S}^{d-1}$ and $x,\, \xi \in \mathbb{R}^d$. We also have a convenient expression for $Q_{+,\frac{ht}{\varepsilon}} b$ in terms of φ_ω,

$$Q_{+,\frac{ht}{\varepsilon}} b\,(x,\xi) = \pi \int_{\mathbb{S}^{d-1}_\omega} \varphi_\omega(x,\xi,\xi^{\cdot 2})\, \mathrm{d}\omega\,.$$

The conclusion is then given by Lemma 6.29. ☐

Lemma 6.29. *For any $\gamma \in (0,1)$, uniformly in $\omega \in \mathbb{S}^{d-1}_\omega$,*

$$\mathcal{N}_k\Big(\int_{\mathbb{R}^+_r} \varphi_\omega(x,\xi,r)\, \kappa^{\zeta}(r - \xi^{\cdot 2})\, \mathrm{d}r - \varphi_\omega(x,\xi,\xi^{\cdot 2})\Big) \leq C_{k,G,\gamma}\, \zeta^{\gamma}\,.$$

Proof. The integral can be expressed as a convolution product

$$\int_{\mathbb{R}_r} \varphi_\omega(x,\xi,r)\, \kappa^{\zeta}(r - \xi^{\cdot 2})\, \mathrm{d}r = \big(\varphi(x,\xi,\cdot) * \kappa^{\zeta}\big)(\xi^{\cdot 2})\,.$$

Since the derivation behaves well with the difference, *i.e.*

$$\begin{aligned} \partial^{\alpha}_x \partial^{\beta}_\xi \left(\big(\varphi_\omega(x,\xi,\cdot) * \kappa^{\zeta}\big)(\xi^{\cdot 2}) - \varphi_\omega(x,\xi,\xi^{\cdot 2})\right) &= \sum_{\alpha',\beta',\gamma'} c_{\alpha',\beta',\gamma'} 2^{|\gamma'|} \xi^{\gamma'} \times \\ &\Big[\Big(\big(\partial^{\alpha'}_x \partial^{\beta'}_\xi \partial^{\gamma'}_r \varphi_\omega\big)(x,\xi,\cdot) * \kappa^{\zeta}\Big)(\xi^{\cdot 2}) - \big(\partial^{\alpha'}_x \partial^{\beta'}_\xi \partial^{\gamma'}_r \varphi_\omega\big)(x,\xi,\xi^{\cdot 2})\Big]\,, \end{aligned}$$

it suffices to apply Lemma 6.30. ☐

For $\zeta > 0$, and $r \in \mathbb{R}$, let $\kappa^{\zeta}(r) = \frac{1}{\pi}\frac{\zeta}{r^2+\zeta^2}$.

Lemma 6.30. *Let f be a function in the Schwartz class. Then for any $\gamma \in (0,1)$, a constant $C_\gamma > 0$ exists such that*

$$\forall \zeta > 0,\ \big\|f * \kappa^{\zeta} - f\big\|_{L^\infty} \leq \max\left\{\|f\|_\infty, \|f'\|_\infty\right\} C_\gamma \zeta^{\gamma}\,.$$

Proof. The formula $f(r_0 + \zeta r) - f(r_0) = \zeta r \int_0^1 f'(r_0 + s\zeta r)\, \mathrm{d}s$ and an interpolation with $|f(r_0 + \zeta r) - f(r_0)| \leq 2\|f\|_\infty$ give for $\gamma \in [0,1]$,

$$|f(r_0 + \zeta r) - f(r_0)| \leq 2 \max\left\{\|f\|_\infty, \|f'\|_\infty\right\} \zeta^{\gamma} |r|^{\gamma}\,.$$

So, for $\gamma \in [0,1)$,

$$\Big|\int_{\mathbb{R}} [f(r_0 + \zeta r) - f(r_0)] \tfrac{\mathrm{d}r}{r^2+1}\Big| \leq \max\{\|f\|_\infty, \|f'\|_\infty\} C_\gamma \zeta^{\gamma}$$

which is the expected result. □

7. Comparisons of the measures of an observable at a mesoscopic scale for the original and approximated dynamics

Remark 7.1. Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$, $\rho \in \mathcal{L}_1 L^2_x$ and $t \geq 0$,

$$m(b, \rho_t^\varepsilon) = \mathrm{Tr}\left[b^W\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \hat{\boldsymbol{\rho}}_t\right],$$
$$m(b, \rho_t^{\varepsilon,app}) = \mathrm{Tr}\left[b^W\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \hat{\boldsymbol{\rho}}_t^{app}\right].$$

Definition 7.2. Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$, $\rho \in \mathcal{L}_1 L^2_x$ a state, $t \geq 0$ and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ we define

$$m(b, \rho, t, \chi) = \mathrm{Tr}\left[\chi\big(\mathrm{d}\Gamma_\varepsilon(\eta)\big)\, b^W\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \chi\big(\mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \hat{\boldsymbol{\rho}}_t\right]$$
$$m^{app}(b, \rho, t, \chi) = \mathrm{Tr}\left[\chi\big(\mathrm{d}\Gamma_\varepsilon(\eta)\big)\, b^W\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \chi\big(\mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \hat{\boldsymbol{\rho}}_t^{app}\right].$$

Proposition 7.3. *Assume $\frac{ht}{\varepsilon}/\sqrt{h}$. Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ non-negative such that $\mathrm{Supp}_\xi\, b \subset B_R \setminus B_{1/R}$ for some $R > 0$, $\rho \in \mathcal{L}_1^+ L^2_x$ with $\mathrm{Tr}\,\rho \leq 1$ and for $j = 1, 2$, $\chi_j \in \mathcal{C}_0^\infty(\mathbb{R}^d_\lambda)$ with values in $[0,1]$, $\chi_j(B_{M_j}) = \{1\}$ for $M_1 = 3R$ and with $\chi_2(\mathbb{R}^d - B_{R+1}) = \{0\}$. There is a constant C_{R,b,χ_1,χ_2} (which does not depend on ρ) such that*

$$m_h^{app}\big(b, (\rho_{\chi_2})_t^{app}\big) - m_h(b, \rho_t) \leq \mathcal{E}_7 = C_{R,b,\chi_1,\chi_2}\big(h + (\tfrac{ht}{\varepsilon}/\sqrt{h})^3 + \mathcal{E}_6\big)$$

with $\rho_{\chi_2} = \chi_2(D_x)\, \rho\, \chi_2(D_x)$.

We use the decomposition $\mathcal{E}_7 = \mathcal{E}_{7.1} + \mathcal{E}_{7.2} + \mathcal{E}_{7.3}$ corresponding to the steps:

1. $m_h(b, \rho_{\chi_2}, t, \chi_1) - m_h(b, \rho_t) \leq \mathcal{E}_{7.1} = Ch$,
2. $m_h^{app}(b, \rho_{\chi_2}, t, \chi_1) - m_h(b, \rho_{\chi_2}, t, \chi_1) \leq \mathcal{E}_{7.2} = C(\frac{ht}{\varepsilon}/\sqrt{h})^3$,
3. $m_h(b, (\rho_{\chi_2})_t^{app}) - m_h^{app}(b, \rho_{\chi_2}, t, \chi_1) \leq \mathcal{E}_{7.3} = \mathcal{E}_6 + Ch$.

7.1. Step 1: Introduction of cutoffs.

We introduce cutoff functions both on the state ρ and the Wick observable $b^W(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta))$.

Proposition 7.4. *Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ non-negative such that $\mathrm{Supp}_\xi\, b \subset B_R$ for some $R > 0$, $\rho \in \mathcal{L}_1^+ L^2_x$, $\mathrm{Tr}\,\rho \leq 1$, and, for $j = 1, 2$, $\chi_j \in \mathcal{C}_0^\infty(\mathbb{R}^d_\lambda)$ with values in $[0,1]$ and $\chi_j(B_{M_j}) = \{1\}$ for some $M_j > 0$. Then there is a constant C_{b,χ_1,χ_2} such that*

$$m(b, \rho_{\chi_2}, t, \chi_1) - m(b, \rho_t) \leq \mathcal{E}_{7.1} = C_{b,\chi_1,\chi_2} h$$

with $\rho_{\chi_2} = \chi_2(D_x) \circ \rho \circ \chi_2(D_x)$.

Proof. Using the functional calculus for the self-adjoint operator $\mathrm{d}\Gamma_\varepsilon(\eta)$ and since

$$\begin{aligned} b(x, \xi - \lambda) &\geq \chi_2(\xi)\, b(x, \xi - \lambda)\, \chi_1(\lambda)\, \chi_2(\xi) \\ &\geq \chi_2(\xi)\, \sharp^h\, b(x, \xi - \lambda)\, \chi_1(\lambda)\, \sharp^h\, \chi_2(\xi) - C_{b,\chi_1,\chi_2} h \end{aligned}$$

holds uniformly in λ, we can write

$$\begin{aligned} &b^W\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big) \\ &\qquad \geq \chi_2(\xi) \circ b^W\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \chi_1(\mathrm{d}\Gamma_\varepsilon(\eta)) \circ \chi_2(\xi) - C_{b,\chi_1,\chi_2} h\,. \end{aligned}$$

And thus

$$\begin{aligned} m(b,\rho_t) &= \operatorname{Tr}\left[b^W\big(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\big)\,\hat{\boldsymbol{\rho}}_t\right] \\ &\geq \operatorname{Tr}\left[b^W\big(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\big)\,\chi_1(\mathrm{d}\Gamma(\eta))\,\widehat{\boldsymbol{\rho}_{\boldsymbol{\chi_2}\,t}}\right] - C_{b,\chi_1,\chi_2}h \end{aligned}$$

since $[H_\varepsilon,\chi_2]=0$. $\square$

7.2. **Step 2: Comparison between truncated solutions.**

Proposition 7.5. *Suppose $\frac{ht}{\varepsilon}\leq\sqrt{h}$. Let $b\in\mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ non-negative, $\rho\in\mathcal{L}_1^+L_x^2$, $\operatorname{Tr}\rho\leq 1$ and $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^d_\lambda)$ with values in $[0,1]$, and $\chi(B_M)=\{1\}$ for some $M>0$, then there is a constant $C_{G,b,\chi}$ such that*

$$|m(b,\rho,t,\chi)-m^{app}(b,\rho,t,\chi)|\leq\mathcal{E}_{7.2}=C_{G,b,\chi}\big(\tfrac{ht}{\varepsilon}/\sqrt{h}\big)^3\,.$$

Set

$$b_\chi = b\big(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\big)\,\chi(\mathrm{d}\Gamma_\varepsilon(\eta))\,. \tag{7.1}$$

We want to control the error when we consider $\operatorname{Tr}[b_\chi\,\boldsymbol{\rho}_t^{app}]$ instead of $\operatorname{Tr}[b_\chi\,\boldsymbol{\rho}_t]$ *i.e.* we want to control $\operatorname{Tr}[b_\chi\,u_t]$ with

$$u_t=\boldsymbol{\rho}_t-\boldsymbol{\rho}_t^{app}\,. \tag{7.2}$$

Since $i\varepsilon\partial_t\boldsymbol{\rho}_t=[H_\varepsilon,\boldsymbol{\rho}_t]$ and $i\varepsilon\partial_t\boldsymbol{\rho}_t^{app}=[H_\varepsilon,\boldsymbol{\rho}_t^{app}]-[H_\varepsilon-H_\varepsilon^{app},\boldsymbol{\rho}_t^{app}]$, the difference u_t is solution of the differential equation

$$i\varepsilon\partial_t u_t=\left[(\xi-\mathrm{d}\Gamma_\varepsilon(\eta))^2,u_t\right]+\left[\Phi_\varepsilon(f_{h,\varepsilon}),u_t\right]-\left[\mathrm{d}\Gamma_\varepsilon(\eta)^2-\varepsilon\,\mathrm{d}\Gamma_\varepsilon(\eta^2),\boldsymbol{\rho}_t^{app}\right]$$

with initial data $u_{t=0}=0$. We can then use the integral expression

$$\operatorname{Tr}[b_\chi\,u_t]=-\tfrac{i}{\varepsilon}\int_0^t\operatorname{Tr}\left[b_\chi i\varepsilon\partial_t u_t\right]\mathrm{d}s\,.$$

Remark 7.6. Let $\mathcal{H}$ be a Hilbert space. If A, $B\in\mathcal{L}(\mathcal{H})$ and $C\in\mathcal{L}_1(\mathcal{H})$, then the cyclicty of the trace gives $\operatorname{Tr}[A\,[B\,,C]]=\operatorname{Tr}[[A\,,B]\,C]$.

Lemma 7.7. *There exists a constant C independent of χ such that for b_χ and u_t defined by Equations (7.1) and (7.2),*

(1) $\left|\frac{1}{\varepsilon}\int_0^t\operatorname{Tr}\left[b_\chi\left[(\xi-\mathrm{d}\Gamma_\varepsilon(\eta))^2\,,u_{h,\varepsilon,s}\right]\right]\mathrm{d}s\right|\leq\frac{h}{\varepsilon}\int_0^t\|u_{h,\varepsilon,s}\|_{\mathcal{L}_1}\,\mathrm{d}s\leq C\frac{h^2t^3}{\varepsilon^3}$,

(2) $\frac{1}{\varepsilon}\int_0^t\operatorname{Tr}\left[b_\chi\left[\mathrm{d}\Gamma_\varepsilon(\eta)^2-\varepsilon\,\mathrm{d}\Gamma_\varepsilon(\eta^2)\,,\boldsymbol{\rho}^{app}\right]\right]\mathrm{d}s=0$,

(3) $\left|\frac{1}{\varepsilon}\int_0^t\operatorname{Tr}\left[b_\chi\left[\Phi_\varepsilon(f_{h,\varepsilon})\,,u_s\right]\right]\mathrm{d}s\right|\leq C\frac{t^3h^{3/2}}{\varepsilon^{7/2}}\Big(\sqrt{\varepsilon}+\sqrt{\frac{t}{2}}\sqrt{\frac{ht}{\varepsilon}}\Big)$.

Proof. For Point 1, let us introduce $\chi_1\succ\chi$ (*i.e.* $\chi_1\in\mathcal{C}_0^\infty$ with values in $[0,1]$ such that $\chi_1\equiv 1$ on $\operatorname{Supp}\chi$) in order to handle only bounded operators:

$$\begin{aligned} &\operatorname{Tr}\left[b_\chi\left[(\xi-\mathrm{d}\Gamma_\varepsilon(\eta))^2,u_s\right]\right] \\ &\quad=\operatorname{Tr}\left[b_\chi\left[\chi_1(\mathrm{d}\Gamma_\varepsilon(\eta))(\xi-\mathrm{d}\Gamma_\varepsilon(\eta))^2,u_s\right]\right] \\ &\quad=\operatorname{Tr}\left[\left[b_\chi\,,\chi_1(\mathrm{d}\Gamma_\varepsilon(\eta))(\xi-\mathrm{d}\Gamma_\varepsilon(\eta))^2\right]u_s\right] \\ &\quad=\operatorname{Tr}\left[\chi(\mathrm{d}\Gamma_\varepsilon(\eta))\,\tfrac{h}{i}\{b(x,\xi),\xi^2\}(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta))\,u_s\right] \\ &\quad=\operatorname{Tr}\left[\tfrac{h}{i}\chi(\mathrm{d}\Gamma_\varepsilon(\eta))\,(2\xi.b)(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta))\,u_s\right]. \end{aligned}$$

The bound $\left\|\chi(\mathrm{d}\Gamma_\varepsilon(\eta))\,(2\xi.b)\big(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\big)\right\|_{\mathcal{L}L^2_\xi\otimes\Gamma L^2_\eta}\le C$ and a time integration bring

$$\Big|\frac{1}{\varepsilon}\int_0^t \mathrm{Tr}\left[b_\chi\big[(\xi-\mathrm{d}\Gamma_\varepsilon(\eta))^2,u_s\big]\right]\mathrm{d}s\Big|\le C\frac{h}{\varepsilon}\int_0^t\|u_s\|_{\mathcal{L}_1L^2_\xi\otimes\Gamma L^2_\eta}\,\mathrm{d}s\,.$$

Then we use that both $\hat{\boldsymbol{\rho}}_t$ and $\hat{\boldsymbol{\rho}}_t^{app}$ have the same initial value $\rho_0\otimes\mathrm{proj}\,\Omega$ with $\rho_0=\sum_j\lambda_j|\psi_{0,j}\rangle\langle\psi_{0,j}|$, $\sum_j\lambda_j=\mathrm{Tr}\,\rho$, $\lambda_j\ge 0$, $\|\psi_{0,j}\|=1$ to write

$$\boldsymbol{\rho}_t=\sum_j\lambda_j|\varphi_{t,j}\rangle\langle\varphi_{t,j}|\,,\qquad \boldsymbol{\rho}_t^{app}=\sum_j\lambda_j|\varphi_{t,j}^{app}\rangle\langle\varphi_{t,j}^{app}|\,,$$

and then $u_t=\sum_j\lambda_j\big(|\Psi_{t,j}-\Psi_{t,j}^{app}\rangle\langle\Psi_{t,j}|-|\Psi_{t,j}^{app}\rangle\langle\Psi_{t,j}^{app}-\Psi_{t,j}|\big)$ and

$$\|u_t\|_{\mathcal{L}_1L^2_\xi}\le 2\sum_j\lambda_j\left\|\Psi_{t,j}-\Psi_{t,j}^{app}\right\|\le C(\tfrac{ht}{\varepsilon}/\sqrt{h})^2\,.$$

This and the integral above yield the result.

For Point 2, let $\chi_1\succ\chi$,

$$\begin{aligned}&\mathrm{Tr}\left[b_\chi\left[\mathrm{d}\Gamma_\varepsilon(\eta)^2-\varepsilon\,\mathrm{d}\Gamma_\varepsilon(\eta^2)\,,u_s\right]\right]\\&\quad=\mathrm{Tr}\left[b_\chi\left[\chi_1(\mathrm{d}\Gamma_\varepsilon(\eta))\left(\mathrm{d}\Gamma_\varepsilon(\eta)^2-\varepsilon\,\mathrm{d}\Gamma_\varepsilon(\eta^2)\right),u_s\right]\right]\\&\quad=\mathrm{Tr}\left[\left[\chi_1(\mathrm{d}\Gamma_\varepsilon(\eta))\left(\mathrm{d}\Gamma_\varepsilon(\eta)^2-\varepsilon\,\mathrm{d}\Gamma_\varepsilon(\eta^2)\right),b_\chi\right]u_s\right]\end{aligned}$$

which vanishes since $\left[\chi_1(\mathrm{d}\Gamma_\varepsilon(\eta))\left(\mathrm{d}\Gamma_\varepsilon(\eta)^2-\varepsilon\,\mathrm{d}\Gamma_\varepsilon(\eta^2)\right),b_\chi\right]=0\,.$

For Point 3, we have, with $\Delta\hat{\Psi}_s=\hat{\Psi}_s-\hat{\Psi}_s^{app}$,

$$\mathrm{Tr}\left[b_\chi[\Phi_\varepsilon(f_{h,\varepsilon})\,,u_s]\right]=\langle\Delta\hat{\Psi}_s|\,[b_\chi\,,\Phi_\varepsilon(f_{h,\varepsilon})]\,|\hat{\Psi}_s\rangle+\langle\hat{\Psi}_s^{app}|\,[b_\chi\,,\Phi_\varepsilon(f_{h,\varepsilon})]\,|\Delta\hat{\Psi}_s\rangle\,.$$

Taking the modulus we obtain

$$\begin{aligned}|\mathrm{Tr}\,[b_\chi\,[\Phi_\varepsilon(f_{h,\varepsilon})\,,u_s]]|\le C\|\Delta\hat{\Psi}_s\|\Big(&\left\|\Phi_\varepsilon(f_{h,\varepsilon})\,\hat{\Psi}_s\right\|+\left\|\Phi_\varepsilon(f_{h,\varepsilon})\,b_\chi\hat{\Psi}_s\right\|\\&+\left\|\Phi_\varepsilon(f_{h,\varepsilon})\,b_\chi^*\hat{\Psi}_s^{app}\right\|+\left\|\Phi_\varepsilon(f_{h,\varepsilon})\,\hat{\Psi}_s^{app}\right\|\Big)\end{aligned}$$

and we observe that

$$\max\left\{\left\|\Phi_\varepsilon(f_{h,\varepsilon})\,\Psi_s^\sharp\right\|,\left\|\Phi_\varepsilon(f_{h,\varepsilon})\,b_\chi\,\hat{\Psi}_s^\sharp\right\|\right\}\le C\left\|f_{h,\varepsilon}\right\|\left\|(\varepsilon+N_\varepsilon)^{1/2}\hat{\Psi}_s^\sharp\right\|$$

and thus, by the number estimate (4) in Proposition 5.3,

$$\Big|\,\mathrm{Tr}\left[b_\chi[\Phi_\varepsilon(f_{h,\varepsilon})\,,u_s]\right]\Big|\le C\|\Delta\hat{\Psi}_s\|\sqrt{\frac{h}{\varepsilon}\|\hat{G}\|_{L^1}}\left(\sqrt{\varepsilon}+\tfrac{s}{\sqrt{2}}\|f_{h,\varepsilon}\|_{L^2_\xi}\right).$$

A time integration gives the result. ∎

7.3. **Step 3: Release of the truncation on the symbol.**

Proposition 7.8. *Let $b\in\mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ non-negative, such that $\mathrm{Supp}_\xi\,b\subset B_R\setminus B_{1/R}$ for some $R>1$, $\rho\in\mathcal{L}_1^+L^2_x$, $\mathrm{Tr}\,\rho\le 1$, with the support of $\hat{\rho}$ in B^2_{R+1} and $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^d_\lambda)$ with values in $[0,1]$, $\chi(B_{3R})=\{1\}$. There is a constant $C_{R,b,\chi}$ such that*

$$m^{app}(b,\rho,t,\chi)-m(b,\rho_t^{app})\ge\mathcal{E}_{7.3}$$

with $\mathcal{E}_{7.3}=\mathcal{E}_6+C_{R,b,\chi}h$, i.e.

$$\mathcal{E}_{7.3}=C\tfrac{ht}{\varepsilon}\left(\tfrac{ht}{\varepsilon}+h+\left[h(\tfrac{ht}{\varepsilon})^{-1}\right]^{d/2-1}+h^{\nu(d,\alpha)}+h^{\gamma\beta(d,\alpha)}\right)+C_{r,b,\chi}h\,.$$

Proof. We restrict the proof to the case of $\rho = |\psi\rangle\,\langle\psi|$ with $\psi \in L^2_x$ since ρ is trace class, then $\hat{\boldsymbol{\rho}}_t = |\hat{\Psi}^{app}_t\rangle\langle\hat{\Psi}^{app}_t|$. We also define a positive symbol $b_1 \in \mathcal{C}^\infty_0(\mathbb{R}^d_\xi)$ such that $\operatorname{Supp} b_1 \subset [R^{-2}, R^2]$ and $b_1(\xi^2) \geq b(x,\xi)$. Then

$$\begin{aligned} &m(b, \rho^{app}_t) - m^{app}(b,\rho,t,\chi) \\ &= \operatorname{Tr}\Big[\,(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta)))^{1/2}\, b^W(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta))\,(1-\chi\,(\mathrm{d}\Gamma_\varepsilon\,(\eta)))^{1/2}\,\hat{\boldsymbol{\rho}}_t\Big] \\ &\leq \operatorname{Tr}\Big[b_1^W\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)\,\big(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta))\big)\,b_1^W\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)\hat{\boldsymbol{\rho}}_t\Big] + \mathcal{O}(h) \end{aligned}$$

with $\hat{\Psi}^{app}_t(\xi) = 1_{[0,M]}(|\xi|)\,\hat{\Psi}^{app}_t(\xi)$ and $\operatorname{Supp} b_1 \subset [R^{-2}, R^2]$. Then we decompose

$$\hat{\Psi}^{app}_t = 1_{[1/2R,2R]}(|\xi|)\,\hat{\Psi}^{app}_t + 1_{[0,M]\setminus[1/2R,2R]}(|\xi|)\,\hat{\Psi}^{app}_t = \hat{\Psi}^{app}_{t,1} + \hat{\Psi}^{app}_{t,2}\,.$$

With $A = b_1^W\big(\,(\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\,\big)\;(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta)))\;b_1^W\big(\,(\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\,\big) \geq 0$ we have the estimate

$$\operatorname{Tr}\big[A\,|\hat{\Psi}^{app}_t\rangle\langle\hat{\Psi}^{app}_t|\big] \leq 2\operatorname{Tr}\big[A\,|\hat{\Psi}^{app}_{t,1}\rangle\langle\hat{\Psi}^{app}_{t,1}|\big] + 2\operatorname{Tr}\big[A\,|\hat{\Psi}^{app}_{t,2}\rangle\langle\hat{\Psi}^{app}_{t,2}|\big]\,.$$

The first term vanishes since

$$\begin{aligned} &\operatorname{Tr}\Big[b_1^W\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)\big(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta))\big)b_1^W\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)\,|\hat{\Psi}^{app}_{t,1}\rangle\langle\hat{\Psi}^{app}_{t,1}|\Big] \\ &\qquad = \operatorname{Tr}\Big[1_{[1/2R,2R]}(|\xi|)\,b_1^W\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)\,(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta))) \\ &\qquad\qquad b_1^W\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)1_{[1/2R,2R]}(|\xi|)\,|\hat{\Psi}^{app}_{t,1}\rangle\langle\hat{\Psi}^{app}_{t,1}|\Big] \end{aligned}$$

and $|\xi| \in [1/2R, 2R]$, $|\xi - \mathrm{d}\Gamma_\varepsilon(\eta)| \leq R$ implies $|\mathrm{d}\Gamma_\varepsilon(\eta)| \leq 3R$ and $\chi(B_{3R}) = \{1\}$. For the second term,

$$\begin{aligned} &\operatorname{Tr}\Big[b_1^W\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)\,(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta)))\,b_1^W\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)|\hat{\Psi}^{app}_{t,2}\rangle\langle\hat{\Psi}^{app}_{t,2}|\Big] \\ &\qquad\qquad \leq \operatorname{Tr}\Big[b_1^W\,\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)|\hat{\Psi}^{app}_{t,2}\rangle\langle\hat{\Psi}^{app}_{t,2}|\Big] \end{aligned}$$

since $1 - \chi(d\Gamma_\varepsilon(\eta)) \leq \mathrm{Id}$. Then we use the computation of the evolution of a symbol of $|\xi|^2$ in the case of the approximated equation as in Remark 6.2 to get that, since $b_1 = b_1(|\xi|^2)$ it is unchanged under the evolution, and

$$\begin{aligned} &\operatorname{Tr}\Big[b_1^W\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)^2|\hat{\Psi}^{app}_{t,2}\rangle\langle\hat{\Psi}^{app}_{t,2}|\Big] \\ &\qquad\qquad \leq \operatorname{Tr}\Big[b_1^W\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)^2|\hat{\psi}_{0,2}\otimes\Omega\rangle\langle\hat{\psi}_{0,2}\otimes\Omega|\Big] + \mathcal{E}_6 \end{aligned}$$

which brings the result observing that

$$\operatorname{Tr}\Big[b_1^W\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)^2|\hat{\psi}_{0,2}\otimes\Omega\rangle\langle\hat{\psi}_{0,2}\otimes\Omega|\Big] = \operatorname{Tr}\Big[b_1^W(\xi^{\cdot 2})^2|\hat{\psi}_{0,2}\otimes\Omega\rangle\langle\hat{\psi}_{0,2}\otimes\Omega|\Big]$$

vanishes since $\operatorname{Supp} b_1 \cap \operatorname{Supp}\hat{\psi}_{0,2} = \emptyset$. □

8. The derivation of the Boltzmann equation for the model

Proposition 8.1. *Let $b \in \mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$ with $\operatorname{Supp}_\xi b \subset B_R \setminus B_{1/R}$. Let ρ a state and $T > 0$ then*

$$\liminf_{h\to 0}\big(m\left(\mathcal{B}^T(T)\,b,\rho\right) - m(b, \rho^h_{N,\Delta t})\big) \leq 0$$

for a fixed $\alpha \in (\frac{3}{4}, 1)$, $\Delta t = \Delta t(h) = h^\alpha$ and $N(h)\,\Delta t(h) = T$.

Lemma 8.2. *With $b_t = e^{tQ}e^{2t\xi.\partial_x}b$, and the hypotheses of Proposition 8.1,*

$$m(b_{\Delta t},\rho) - m(b,\rho^h_{\Delta t}) \\ \leq C\big(h + (\Delta t/\sqrt{h})^3 + (\Delta t/\sqrt{h})^4 + \Delta t\big(\Delta t + h + (h/\Delta t)^{\frac{d}{2}-1} + h^\mu\big)\big)\,.$$

Proof. We recall that $\rho^h_{\Delta t} = \rho^\varepsilon_{\varepsilon\Delta t/h}$ so that with $\frac{ht}{\varepsilon} = \Delta t$, from Section 7,

$$\begin{aligned} m\big(b,(\rho_{\chi_2})^{h,app}_{\Delta t}\big) &- m(b,\rho^h_{\Delta t}) \\ &= m\big(b,(\rho_{\chi_2})^{\varepsilon,app}_{t}\big) - m(b,\rho^\varepsilon_t) \\ &\leq C\left(h + \left(\tfrac{ht}{\varepsilon}/\sqrt{h}\right)^3 + \left(\tfrac{ht}{\varepsilon}/\sqrt{h}\right)^4 + \tfrac{ht}{\varepsilon}\left(\tfrac{ht}{\varepsilon} + h + (\varepsilon/t)^{d/2-1} + h^\mu\right)\right) \\ &\leq C\Big(h + (\Delta t/\sqrt{h})^3 + (\Delta t/\sqrt{h})^4 + \Delta t\big(\Delta t + h + (h/\Delta t)^{d/2-1} + h^\mu\big)\Big) \end{aligned}$$

and from Section 6 also used with $\frac{ht}{\varepsilon} = \Delta t$ we get

$$m(b_t,\rho_{\chi_2}) - m\big(b,(\rho_{\chi_2})^{\varepsilon,app}_t\big) \leq \mathcal{E}_6 \leq \mathcal{E}_7$$

and this term will be in particular controlled if we control the previous one. Finally from the conservation of the support in ξ of the symbol by the approximated Boltzmann equation we get

$$m(b_t,\rho) - m(b_t,\rho_{\chi_2}) \leq \mathcal{O}(h^\infty)$$

for χ_2 a cutoff function chosen so that $\chi_2(B_R) = \{1\}$.

Thus we fix, for $j = 1,2$, two cutoff functions $\chi_j \in \mathcal{C}^\infty_0(\mathbb{R}^d_\lambda)$ with values in $[0,1]$, $\chi_j(B_{M_j}) = \{1\}$ for $M_1 = 3R$ and $M_2 = 1$ and with $\chi_2(\mathbb{R}^d \setminus B_{R+1}) = \{0\}$. ☐

Proof of Propostition 8.1. Let, for $k \in \mathbb{N}$, $\Delta t > 0$, $b_{k,\Delta t} = \left(e^{\Delta tQ}e^{2\Delta t\xi.\partial_x}\right)^k b$. Iterating the estimation of the Lemma $N(h)$ times brings

$$m\big(b_{N,\Delta t},\rho\big) - m\big(b,\rho^\varepsilon_{N(h),\varepsilon\Delta t/h}\big) \\ \leq CN\Big(h + (\Delta t/\sqrt{h})^3 + (\Delta t/\sqrt{h})^4 + \Delta t\big(\sqrt{\Delta t} + h + (h/\Delta t)^{\frac{d}{2}-1} + h^\mu\big)\Big)$$

with $N\Delta t = T$ and $h^\alpha \leq \frac{ht}{\varepsilon} = \Delta t \leq 1$ for some $\alpha \in (1/2,1)$. Thus we can choose $\Delta t = \frac{th}{\varepsilon} = h^\alpha$ and thus $N = Th^{-\alpha}$. Then we get the estimate

$$\begin{aligned} m\big(b_{N,\Delta t},\rho\big) &- m\big(b,\rho_{N,\varepsilon\Delta t/h}\big) \\ &\leq CTh^{-\alpha}\big(h + h^{3\alpha-3/2} + h^{4\alpha-2} + h^\alpha(h^{\alpha/2} + h + h^{(1-\alpha)(d/2-1)} + h^\mu)\big) \\ &\leq CT\, o_{h\to 0}(1)\,, \end{aligned}$$

for $\alpha \in (\frac{3}{4},1)$. Finally it suffices to prove that

$$\lim_{h\to 0} m\big(b_{N(h),\Delta t(h)},\rho\big) = m(b_T,\rho)$$

which is true since the estimates of Proposition 3.7 prove that, for some constant $C > 0$, $\|b_{N,\Delta t} - b_T\|_{\mathcal{L}L^2_x} \leq \frac{C}{N}$. ☐

Acknowledgement. We would like to thank Francis Nier and Zied Ammari for very helpful discussions, and many remarks and comments on this paper.

References

[1] Zied Ammari and Francis Nier. Mean field limit for bosons and infinite dimensional phase-space analysis. *Ann. Henri Poincaré*, 9(8):1503–1574, 2008.
[2] Zied Ammari and Francis Nier. Mean field limit for bosons and propagation of Wigner measures. *J. Math. Phys.*, 50(4):042107, 16, 2009.
[3] Stéphane Attal and Alain Joye. Weak coupling and continuous limits for repeated quantum interactions. *J. Stat. Phys.*, 126(6):1241–1283, 2007.
[4] Stéphane Attal and Yan Pautrat. From repeated to continuous quantum interactions. *Ann. Henri Poincaré*, 7(1):59–104, 2006.
[5] Guillaume Bal, George Papanicolaou, and Leonid Ryzhik. Radiative transport limit for the random Schrödinger equation. *Nonlinearity*, 15(2):513–529, 2002.
[6] Philippe Bechouche, Frédéric Poupaud, and Juan Soler. Quantum transport and Boltzmann operators. *J. Stat. Phys.*, 122(3):417–436, 2006.
[7] Feliks A. Berezin. *The method of second quantization*. Academic Press, New York, 1966.
[8] Carlo Boldrighini, Leonid A. Bunimovich, and Yakov G. Sinaĭ. On the Boltzmann equation for the Lorentz gas. *J. Statist. Phys.*, 32(3):477–501, 1983.
[9] Ola Bratteli and Derek W. Robinson. *Operator algebras and quantum statistical mechanics. 2*. Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1997. Equilibrium states. Models in quantum statistical mechanics.
[10] Nicolas Burq. Mesures semi-classiques et mesures de défaut. *Astérisque*, (245):Exp. No. 826, 4, 167–195, 1997. Séminaire Bourbaki, Vol. 1996/97.
[11] Thomas Chen. Localization lengths and Boltzmann limit for the Anderson model at small disorders in dimension 3. *J. Stat. Phys.*, 120(1-2):279–337, 2005.
[12] Robert Dautray and Jacques-Louis Lions. *Analyse mathématique et calcul numérique pour les sciences et les techniques. Tome 1*. Collection du Commissariat à l'Énergie Atomique: Série Scientifique. [Collection of the Atomic Energy Commission: Science Series]. Masson, Paris, 1984.
[13] Robert Dautray and Jacques-Louis Lions. *Analyse mathématique et calcul numérique pour les sciences et les techniques. Tome 3*. Collection du Commissariat à l'Énergie Atomique: Série Scientifique. [Collection of the Atomic Energy Commission: Science Series]. Masson, Paris, 1985.
[14] László Erdős, Manfred Salmhofer, and Horng-Tzer Yau. Quantum diffusion of the random Schrödinger evolution in the scaling limit. II. The recollision diagrams. *Comm. Math. Phys.*, 271(1):1–53, 2007.
[15] László Erdős, Manfred Salmhofer, and Horng-Tzer Yau. Quantum diffusion of the random Schrödinger evolution in the scaling limit. *Acta Math.*, 200(2):211–277, 2008.
[16] László Erdös and Horng-Tzer Yau. Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation. *Comm. Pure Appl. Math.*, 53(6):667–735, 2000.
[17] Gerald B. Folland. *Quantum field theory*, volume 149 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008. A tourist guide for mathematicians.
[18] Giovanni Gallavotti. Divergences and the approach to equilibrium in the lorentz and the wind-tree models. *Phys. Rev.*, 185(1):308–322, Sep 1969.
[19] Patrick Gérard. Mesures semi-classiques et ondes de Bloch. In *Séminaire sur les Équations aux Dérivées Partielles, 1990–1991*, pages Exp. No. XVI, 19. École Polytech., Palaiseau, 1991.
[20] Patrick Gérard. Microlocal defect measures. *Comm. Partial Differential Equations*, 16(11):1761–1794, 1991.
[21] Patrick Gérard, Peter A. Markowich, Norbert J. Mauser, and Frédéric Poupaud. Homogenization limits and Wigner transforms. *Comm. Pure Appl. Math.*, 50(4):323–379, 1997.
[22] Patrick Gérard, Peter A. Markowich, Norbert J. Mauser, and Frédéric Poupaud. Erratum: "Homogenization limits and Wigner transforms" [Comm. Pure Appl. Math. **50** (1997), no. 4, 323–379; MR1438151 (98d:35020)]. *Comm. Pure Appl. Math.*, 53(2):280–281, 2000.
[23] Jean Ginibre and Giorgio Velo. The classical field limit of scattering theory for nonrelativistic many-boson systems. I. *Comm. Math. Phys.*, 66(1):37–76, 1979.
[24] Jean Ginibre and Giorgio Velo. The classical field limit of scattering theory for nonrelativistic many-boson systems. II. *Comm. Math. Phys.*, 68(1):45–68, 1979.

[25] Jean Ginibre and Giorgio Velo. The classical field limit of nonrelativistic bosons. I. Borel summability for bounded potentials. *Ann. Physics*, 128(2):243–285, 1980.

[26] Jean Ginibre and Giorgio Velo. The classical field limit of nonrelativistic bosons. II. Asymptotic expansions for general potentials. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 33(4):363–394, 1980.

[27] James Glimm and Arthur Jaffe. *Quantum physics*. Springer-Verlag, New York, second edition, 1987. A functional integral point of view.

[28] Klaus Hepp. The classical limit for quantum mechanical correlation functions. *Comm. Math. Phys.*, 35:265–277, 1974.

[29] Ting-Guo Ho, Lawrence J. Landau, and A. J. Wilkins. On the weak coupling limit for a Fermi gas in a random potential. *Rev. Math. Phys.*, 5(2):209–298, 1993.

[30] Pierre-Louis Lions and Thierry Paul. Sur les mesures de Wigner. *Rev. Mat. Iberoamericana*, 9(3):553–618, 1993.

[31] André Martinez. *An introduction to semiclassical and microlocal analysis*. Universitext. Springer-Verlag, New York, 2002.

[32] Frédéric Poupaud and Alexis Vasseur. Classical and quantum transport in random media. *J. Math. Pures Appl. (9)*, 82(6):711–748, 2003.

[33] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[34] Michael Reed and Barry Simon. *Methods of modern mathematical physics. III*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1979. Scattering theory.

[35] Barry Simon. *The $P(\phi)_2$ Euclidean (quantum) field theory*. Princeton University Press, Princeton, N.J., 1974. Princeton Series in Physics.

[36] Herbert Spohn. Derivation of the transport equation for electrons moving through random impurities. *J. Statist. Phys.*, 17(6):385–412, 1977.

[37] Herbert Spohn. Kinetic equations from hamiltonian dynamics: Markovian limits. *Rev. Mod. Phys.*, 52(3):569–615, Jul 1980.

IRMAR, UMR-CNRS 6625, Université de Rennes 1, campus de Beaulieu, 35042 Rennes Cedex, France. ENS de Cachan, Antenne de Bretagne, Campus de Ker Lann, Av. R. Schuman, 35170 Bruz, France.

E-mail address: sebastien.breteaux@ens-cachan.org